Fibonacci and Golden Ratio Formulae

Here are almost 300 formula involving the Fibonacci numbers and the golden ratio together with the Lucas numbers and the General Fibonacci series (the G series). This forms a major reference page for Ron Knott's Fibonacci Web site (http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/) where there are many more details and explanations with applications, puzzles and investigations aimed at secondary school students and teachers as well as interested mathematical enthusiasts.
Note that it is **easy to search for a named formula** on this page since it is an HTML page and the formulae are not images. In your browser main menu, under the **Edit** menu look for **Find...** and type Vajda-*N* or Dunlap-*N* for the relevant formula. Full references are at the foot of this document.

> **You can freely change Text Size using the menu items for the browser you are using to view this page in order to increase the size of the symbols and formulae on this web page.**

Contents of This Page

- **Definitions and Notation**
- **Linear Formulae**
  - Two Fibonacci numbers
  - Two Lucas numbers
  - Sums with a Fibonacci and a Lucas number
  - Golden Ratio Formulae
    - Basic Phi Formulae
    - Golden Ratio with Fibonacci and Lucas
- **Order 2 Formulae**
  - Fibonacci numbers
  - Lucas numbers
  - Fibonacci and Lucas Numbers
- **Fibonacci and Lucas Factors**
- **Higher Order Fibonacci and Lucas**
  - Fibonacci and Lucas cubed
  - Fibonacci and Lucas to the fourth
  - Fibonacci and Lucas Higher Powers
- **G Formulae**
  - Basic G Formulae
  - G Formulae of Order 2 or more
- **Summations**
  - Fibonacci and Lucas Summations
  - Decimal (and other bases) fractions
  - Summations with fractions
  - Order 2 summations
  - Summations of order > 2
  - G Summations
  - Summations with Binomial Coefficients
  - Powers of Fibonacci and Lucas as Sums
  - Summations with Binomial and G Series
- **Trigonometric Formulae**
- **Hyperbolic Functions**
- **Complex Numbers**

- Generating Functions
- References

Definitions and Notation

Beware of different golden ratio symbols used by different authors!
Where a formula below (or a simple re-arrangement of it) occurs in either Vajda or Dunlap's book, the reference number they use is given here. Dunlap's formulae are listed in his Appendix A3. Hoggatt's formula are from his "Fibonacci and Lucas Numbers" booklet. Full bibliographic details are at the end of this page in the References section.

As used here	**Vajda**	**Dunlap**	**Knuth**	**Definition**	**Description**
Phi Φ	τ	τ	ϕ, α	$\frac{\sqrt{5}+1}{2} = 1.6180339...$	*Koshy uses* α *(page 78)*
phi ϕ	$-\sigma$	$-\phi$	$-\beta$	$\frac{\sqrt{5}-1}{2} = 0.6180339...$	*Koshy uses* $-\beta$ *(page 78)*
abs(x) $\lvert x \rvert$	$\lvert x \rvert$	$\lvert x \rvert$	$\lvert x \rvert$	*abs(x) = x if x≥0; abs(x) = –x if x<0*	*the absolute value of a number, its magnitude; ignore the sign;*
floor(x) $\lfloor x \rfloor$	*[x]*	*trunc(x), not used for x<0*	$\lfloor x \rfloor$	*the nearest integer ≤ x.*	***When x>0***, *this is "the integer part of x" or "truncate x" i.e. delete any fractional part after the decimal point. 3=floor(3)=floor(3.1)=floor(3.9), -4=floor(-4)=floor(-3.1)=floor(-3.9)*
round(x) [x]	$[\,x + \frac{1}{2}\,]$	*trunc(x + 1/2)*		*the nearest integer to x; trunc(x+0.5)*	*3=round(3)=round(3.1), 4=round(3.9), -4=round(-4)=round(-3.9), -3=round(-3.1) 4=round(3.5), -3=round(-3.5)*
ceil(x) $\lceil x \rceil$	-	-	$\lceil x \rceil$	*the nearest integer ≥ x.*	*3=ceil(3), 4=ceil(3.1)=ceil(3.9), -3=ceil(-3)=ceil(-3.1)=ceil(-3.9)*
fract(x) frac(x)	-	-	***x mod 1***	*x – floor(x)*	*the fractional part of x, i.e. the part of abs(x)* ***after*** *the decimal point*
$\binom{n}{r}$	$\binom{n}{r}$	$\binom{n}{r}$	$\binom{n}{r}$	$\frac{n!}{r!\,(n-r)!}$	${}_nC_r$*; n choose r; the element in row n column r of Pascal's Triangle; the coefficient of* x^r *in* $(1+x)^n$*; the number of ways of choosing r objects from a set of n different objects. n≥0 and r≥0.*

Fibonacci-type series with the rule S(i)=S(i-1)+S(i-2) for all integers i:

i	...	–6	–5	–4	–3	–2	–1	0	1	2	3	4	5	6	...
Fibonacci F(i)	...	–8	5	–3	2	–1	1	0	1	1	2	3	5	8	...
Lucas L(i)	...	18	–11	7	–4	3	–1	2	1	3	4	7	11	18	...

General Fib G(a,b,i)	...	13a–8b	–8a+5b	5a–3b	–3a+2b	2a–b	–a+b	**a**	**b**	a+b	a+2b	2a+3b	3a+5b	5a+8b	...

Formula	*Refs*	*Comments*
$F(0) = 0, F(1) = 1, F(n+2) = F(n + 1) + F(n)$	-	Definition of the Fibonacci series
$F(-n) = (-1)^{n+1} F(n)$	Vajda-2, Dunlap-5	Extending the Fibonacci series 'backwards'
$L(0) = 2, L(1) = 1, L(n + 2) = L(n + 1) + L(n)$	-	Definition of the Lucas series
$L(-n) = (-1)^n L(n)$	Vajda-4, Dunlap-6	Extending the Lucas series 'backwards'
$G(n + 2) = G(n + 1) + G(n)$	Vajda-3, Dunlap-4	Definition of the Generalised Fibonacci series, G(0) and G(1) needed
$Phi = 1.618... = \frac{\sqrt{5} + 1}{2}$	Dunlap-63	Phi and –phi are the roots of $x^2 = x + 1$
$phi = 0.618... = \frac{\sqrt{5} - 1}{2}$	Dunlap-65	Beware! Dunlap occasionally uses ϕ to represent our phi = 0.61803.., but more frequently he uses ϕ to represent –0.61803.. !
$2 F(n + 1) = F(n) + \sqrt{(5 F(n)^2 + 4(-1)^n)}$		F(n+1) from F(n): Problem B-42, S Basin, FQ, 2 (1964) page 329

Linear Formulae

Linear relationships involve only sums or differences of Fibonacci numbers or Lucas numbers or their multiples.

Linear Sums of Fibonacci numbers

$F(n + 2) + F(n) + F(n - 2) = 4 F(n)$	B&Q(2003)-Identity 18
$F(n + 2) + F(n) = L(n + 1)$	by Definition of L(n)
$F(n + 2) - F(n) = F(n + 1)$	by Definition of F(n)
$F(n + 3) + F(n) = 2 F(n + 2)$	B&Q(2003)-Identity 16
$F(n + 3) - F(n) = 2 F(n + 1)$	-
$F(n + 4) + F(n) = 3 F(n + 2)$	B&Q(2003)-Identity 17
$F(n + 2) + F(n - 2) = 3 F(n)$	B&Q(2003)-Identity 7
$F(n + 2) - F(n - 2) = L(n)$	Hoggatt-I14
$F(n + 4) - F(n) = L(n + 2)$	-
$F(n + 5) + F(n) = F(n + 2) + L(n + 3)$	-
$F(n + 5) - F(n) = L(n + 2) + F(n + 3)$	-
$F(n + 6) + F(n) = 2 L(n + 3)$	-

$F(n+6) - F(n) = 4\ F(n+3)$	-
$F(n+1) + F(n-1) = L(n)$	Vajda-6, Hoggatt-I8, Dunlap-14, Koshy-5.14
$F(n) + 2\ F(n-1) = L(n)$	(Dunlap-32)
$F(n+2) - F(n-2) = L(n)$	Vajda-7a, Dunlap-15, Koshy-5.15
$F(n+3) - 2\ F(n) = L(n)$	possible correction for Dunlap-31
$F(n+2) - F(n) + F(n-1) = L(n)$	possible correction for Dunlap-31
$F(n) + F(n+1) + F(n+2) + F(n+3) = L(n+3)$	C Hyson(*)

Linear Sums of Lucas numbers

$L(n-1) + L(n+1) = 5\ F(n)$	Vajda-5, Dunlap-13, Koshy-5.16, B&Q(2003)-Identity 34, Hoggatt-I9
$L(n) + L(n+3) = 2\ L(n+2)$	-
$L(n) + L(n+4) = 3\ L(n+2)$	-
$2\ L(n) + L(n+1) = 5\ F(n+1)$	B&Q(2003)-Identity 52
$L(n+2) - L(n-2) = 5\ F(n)$	-
$L(n+3) - 2\ L(n) = 5\ F(n)$	-

Linear Sum of a Fibonacci and a Lucas number

$F(n) + L(n) = 2\ F(n+1)$	Vajda-7b, Dunlap-16, B&Q-Identity 51
$L(n) + 5\ F(n) = 2\ L(n+1)$	-
$3\ F(n) + L(n) = 2\ F(n+2)$	Vajda-26, Dunlap-28
$3\ L(n) + 5\ F(n) = 2\ L(n+2)$	Vajda-27, Dunlap-29

Golden Ratio Formulae

$$Phi = \frac{\sqrt{5} + 1}{2};\ phi = \frac{\sqrt{5} - 1}{2}$$

Basic Phi Formulae

$Phi\ phi = 1$	Vajda page 51(3), Dunlap-65
$Phi + phi = \sqrt{5}$	-
$Phi\ /\ phi = Phi + 1$	-
$phi\ /\ Phi = 1 - phi$	-
$Phi - phi = 1$	-

$Phi = phi + 1 = \sqrt{5} - phi$	-
$phi = Phi - 1 = \sqrt{5} - Phi$	-
$Phi^2 = 1 + Phi$	Vajda page 51(4), Dunlap-64
$phi^2 = 1 - phi$	Vajda page 51(4), Dunlap-64
$Phi^{n+2} = Phi^{n+1} + Phi^n$	-
$(-phi)^{n+2} = (-phi)^{n+1} + (-phi)^n$	-
$phi^n = phi^{n+1} + phi^{n+2}$	-
$(-Phi)^n = (-Phi)^{n+1} + (-Phi)^{n+2}$	-

Golden Ratio with Fibonacci and Lucas

$\lim_{n\to\infty} \frac{F(n+1)}{F(n)} = Phi$	Vajda-101
$\lim_{n\to\infty} \frac{F(n+m)}{F(n)} = Phi^m$	Vajda-101a
$F(n) = \frac{Phi^n - (-phi)^n}{\sqrt{5}}$	**"Binet's" Formula** De Moivre(1718), Binet(1843), Lamé(1844), Vajda-58, Dunlap-69, Hoggatt-page 11, B&Q(2003)-Identity 240
$L(n) = Phi^n + (-phi)^n$	Vajda-59, Dunlap-70, B&Q(2003)-Identity 241
$F(n) = round\left(\frac{Phi^n}{\sqrt{5}}\right), if\ n\ge 0$	Vajda-62, Dunlap-71 corrected, B&Q(2003)-Identity 240 Corollary 30
$L(n) = round(Phi^n), if\ n\ge 2$	Vajda-63, Dunlap-72, B&Q(2003)-Corollary 35
$F(-n) = round\left(\frac{-(-phi)^{-n}}{\sqrt{5}}\right), if\ n\ge 0$	-
$L(-n) = round(\ (-phi)^{-n}\ ),\ n\ge 3$	-
$F(-n) = (-1)^{n+1} round\left(\frac{Phi^n}{\sqrt{5}}\right), if\ n\ge 0$	-
$F(n+1) = round(Phi\ F(n)), if\ n\ge 2$	Vajda-64, Dunlap-73
$L(n+1) = round(Phi\ L(n)), if\ n\ge 4$	Vajda-65, Dunlap-74
$fract(\ F(2n)\ phi\ ) = 1 - phi^{2n}$	Knuth vol 1, Ex 1.2.8 Qu 31
$fract(\ F(2n+1)\ Phi\ ) = phi^{2n-1}$	Knuth vol 1, Ex 1.2.8 Qu 31
$Phi^n = \frac{L(n) + F(n)\sqrt{5}}{2}$	Rabinowitz-25, B&Q(2003)-Identity 242, Vajda page 125
$Phi^n = Phi\ F(n) + F(n-1)$	Rabinowitz-28, B&Q(2003)-Corrolary 33
$Phi^n = F(n+1) + F(n)\ phi$	Rabinowitz-28, B&Q(2003)-Corollary 33

$(-phi)^n = \frac{L(n) - F(n)\sqrt{5}}{2}$	I Ruggles (1963) FQ 1.2 pg 80, Rabinowitz-25, B&Q(2003)-Identity 243, Vajda page 125
$Phi^n = \frac{L(n) + F(n)\sqrt{5}}{2}$	I Ruggles (1963) FQ 1.2 pg 80
$(-phi)^n = -phi\ F(n) + F(n-1)$	Rabinowitz-28
$(-phi)^n = F(n+1) - Phi\ F(n)$	Vajda-103b, Dunlap-75
$\sqrt{5}\ Phi^n = Phi\ L(n) + L(n-1)$	-
$\sqrt{5}\ (-phi)^n = phi\ L(n) - L(n-1)$	-

Order 2 Formulae

Order 2 means these formulae have terms involving the *product of at most 2* Fibonacci or Lucas numbers.

Fibonacci numbers

$F(n)^2 + 2\ F(n-1)F(n) = F(2n)$	-
$F(n+1)^2 + F(n)^2 = F(2n+1)$	Vajda-11, Dunlap-7, Lucas(1878), B&Q(2003)-Identity 13, Hoggatt-I11
$F(n+1)^2 - F(n-1)^2 = F(2n)$	Lucas(1878), B&Q(2003)-Identity 14, Hoggatt-I10
$F(n+1)^2 - F(n)^2 = F(n+2)\ F(n-1)$	Vajda-12, Dunlap-8
$F(n+2)^2 = 3\ F(n+1)^2 - F(n)^2 -(-1)^n$	V E Hoggatt B-208 FQ 9 (1971) pg 217 and 220.
$F(n+3)^2 + F(n)^2 = 2\ (\ F(n+1)^2 + F(n+2)^2\ )$	B&Q(2003)-Identity 30
$F(n+k+1)^2 + F(n-k)^2 = F(2k+1)F(2n+1)$	a generalization of Vajda-11,Dunlap-7 Melham(1999)
$F(\ n+p\ )^2 - F(\ n-p\ )^2 = F(\ 2n\ )F(\ 2p\ )$	I Ruggles (1963) FQ 1.2 pg 77; Hoggatt-I25
$F(n+1)\ F(n-1) - F(n)^2 = (-1)^n$	**Cassini's Formula**(1680), Simson(1753), Vajda-29, Dunlap-9, Hoggatt-I13 special case of Catalan's Identity with r=1 B&Q(2003)-Identity 8
$F(n)^2 - F(n+r)F(n-r) = (-1)^{n-r}F(r)^2$	**Catalan's Identity**(1879)
$F(n)F(m+1) - F(m)F(n+1) = (-1)^m F(n-m)$	**d'Ocagne's Identity**, special case of Vajda-9 with G=F
$F(n+m) = F(n+1)F(m+1) - F(n-1)F(m-1)$	B&Q(2003)-Identity 231
$F(n+m) = F(m)\ F(n+1) + F(m-1)\ F(n)$	alternative to Dunlap-10, B&Q(2003)-Identity 3; variation of R T Hansen FQ (1972) "Generating Identities for Fibonacci and Lucas Triples" p 571-578
$F(n) = F(m)\ F(n+1-m) + F(m-1)\ F(n-m)$	I Ruggles (1963) FQ 1.2 pg 79; Dunlap-10, special case of Vajda-8

$F(n)\ F(n+1) = F(n-1)\ F(n+2) + (-1)^{n-1}$	Vajda-20a special case: i:=1;k:=2;n:=n-1; Hoggatt-I19
$F(n+i)\ F(n+k) - F(n)\ F(n+i+k) = (-1)^n\ F(i)\ F(k)$	Vajda-20a=Vajda-18 (corrected) with G:=H:=F
$F(a)F(b) - F(c)F(d) = (-1)^r(F(a-r)F(b-r) - F(c-r)F(d-r))$ *a+b=c+d for any integers a,b,c,d,r*	Johnson FQ 42 (2004) B-960 'A Fibonacci Iddentity', solution pg 90 also Johnson-7 Cassini, Catalan and D'Ocagne's Identities are all special cases of this formula
$(F(n-1)F(n+2))^2 + (2\ F(n)F(n+1))^2 = (F(n+1)F(n+2) - F(n-1)F(n))^2 = F(2n+1)^2$	A F Horadam *FQ* 20 (1982) pgs 121-122, B&Q(2003)-Identity 19 (corrected) special case of **Generalised Fibonacci Pythagorean Triples**

F(nk) is a multiple of F(n)	B&Q(2003)-Theorem 1, Vajda Theorem I page 82
$gcd(F(m),F(n)) = F(gcd(m,n))$	Lucas (1878) B&Q(2003)-Theorem 6,Vajda Theorem II page 83
$F(mn+r) \equiv \pm F(r)\ (mod\ F(n)\ )$	Knuth Vol 1 Ex 1.2.8 Qu. 32, Vajda page 86

Lucas numbers

$L(n+2)^2 = 3\ L(n+1)^2 - L(n)^2 + 10(-1)^n$	V E Hoggatt B-208 FQ 9 (1971) pg 217 and 220.
$L(n+2)\ L(n-1) = L(n+1)^2 - L(n)^2$	-
$L(n+1)\ L(n-1) - L(n)^2 = -5\ (-1)^n$	B&Q(2003)-Identity 60
$L(2n) + 2\ (-1)^n = L(n)^2$	Vajda-17c, Dunlap-12, B&Q(2003)-Identity 36
$L(n+m) + (-1)^m\ L(n-m) = L(m)\ L(n)$	Vajda-17a, Dunlap-11 (special cases: Hoggatt-I15,I18)
$L(4n) + 2 = L(2n)^2$	Hoggatt-I15, special case of Vajda-17a
$2\ L(n+1) = L(n) + \sqrt{5}\ \sqrt{(L(n)^2 - 4(-1)^n)}$	L(n+1) from L(n): Problem B-42, S Basin, FQ 2 (1964) page 329

$gcd(L(m),L(n)) = L(gcd(m,n))$, *if both s/d and t/d are odd integers*	Vajda page 86
$L(mn+r) \equiv \pm L(r)\ (mod\ L(n)\ )$	(Vajda page 87)

Fibonacci and Lucas Numbers

$F(2n) = F(n)\ L(n)$	Vajda-13, Hoggatt-I7, Koshy-5.13, B&Q(2003)-Identity 33
$5\ F(n) = L(n+1) + L(n-1)$	
$L(n+1)^2 + L(n)^2 = 5\ F(2n+1)$	Vajda-25a
$L(n+1)^2 - L(n-1)^2 = 5\ F(2n)$	(corrected 29June09)
$L(n+1)^2 - 5\ F(n)^2 = L(2n+1)$	(corrected 29June09)
$L(2n) - 2\ (-1)^n = 5\ F(n)^2$	Vajda-23, Dunlap-25

$L(n)^2 - 4(-1)^n = 5\ F(n)^2$	B&Q(2003)-Identity 53, Hoggatt-I12
$F(n+k) + F(n-k) = F(n)L(k)$, k even;	Bergum and Hoggatt (1975) equn (5)
$F(n+k) + F(n-k) = L(n)F(k)$, k odd;	Bergum and Hoggatt (1975) equn (6)
$F(n+k) - F(n-k) = F(n)L(k)$, k odd;	Bergum and Hoggatt (1975) equn (7)
$F(n+k) - F(n-k) = L(n)F(k)$, k even;	Bergum and Hoggatt (1975) equn (8)
$L(n+k) + L(n-k) = L(n)L(k)$, k even	Bergum and Hoggatt (1975) equn (9)
$L(n+k) + L(n-k) = 5F(n)F(k)$, k odd	Bergum and Hoggatt (1975) equn (10)
$L(n+k) - L(n-k) = L(n)L(k)$, k odd	Bergum and Hoggatt (1975) equn (11)
$L(n+k) - L(n-k) = 5F(n)F(k)$, k even	Bergum and Hoggatt (1975) equn (12)
$F(n + 1)\ L(n) = F(2n + 1) + (-1)^n$	Vajda-30, Vajda-31, Dunlap-27, Dunlap-30
$L(n + 1)\ F(n) = F(2n + 1) - (-1)^n$	-
$F(2n + 1) = F(n + 1)\ L(n + 1) - F(n)\ L(n)$	Vajda-14, Dunlap-18
$L(2n + 1) = F(n + 1)\ L(n + 1) + F(n)\ L(n)$	-
$L(4n + 1) = 1 + L(2n + 1)\ F(2n)$	Hoggatt?
$L(4n + 3) = 1 + L(2n + 1)\ F(2n + 2)$	Hoggatt?
$L(m)\ L(n) + L(m - 1)\ L(n - 1) = 5\ F(m + n - 1)$	R T Hansen FQ (1972) "Generating Identities for Fibonacci and Lucas Triples" p 571-578
$L(n)^2 - 2\ L(2n) = -5\ F(n)^2$	Vajda-22, Dunlap-24
$5\ F(n)^2 - L(n)^2 = 4\ (-1)^{n+1}$	Vajda-24, Dunlap-26
$F(n)^2 + L(n)^2 = = 4\ F(n + 1)^2 - 2\ F(2n)$	FQ (2003)vol 41, B-936, M A Rose, page 87
$5\ (F(n)^2 + F(n + 1)^2) = L(n)^2 + L(n + 1)^2$	Vajda-25
$F(n)\ L(m) = F(n + m) + (-1)^m\ F(n - m)$	a recurrence relation for $F(n+km)$: Vajda-15a, Dunlap-19
$L(n)\ F(m) = F(n + m) - (-1)^m\ F(n - m)$	Vajda-15b, Dunlap-20
$5\ F(m)\ F(n) = L(n + m) - (-1)^m\ L(n - m)$	Vajda-17b, Dunlap-23, (special cases:Hoggatt-I16,I17)
$2\ F(n + m) = L(m)\ F(n) + L(n)\ F(m)$	Vajda-16a, Dunlap-2, FQ (1967) B106 H H Ferns pp 466-467
$2\ L(n + m) = L(m)\ L(n) + 5\ F(n)\ F(m)$	FQ (1967) B106 H H Ferns pp 466-467
$F(m)\ L(n) + F(m - 1)\ L(n - 1) = L(m + n - 1)$	R T Hansen FQ (1972) "Generating Identities for Fibonacci and Lucas Triples" p 571-578
$(-1)^m\ 2\ F(n - m) = L(m)\ F(n) - L(n)\ F(m)$	Vajda-16b, Dunlap-22
$L(n + i)\ F(n + k) - L(n)\ F(n + i + k) = (-1)^{n+1}\ F(i)\ L(k)$	Vajda-19a
$F(n + i)\ L(n + k) - F(n)\ L(n + i + k) = (-1)^n\ F(i)\ L(k)$	Vajda-19b
$L(n + i)\ L(n + k) - L(n)\ L(n + i + k) = (-1)^{n+1}\ 5\ F(i)\ F(k)$	Vajda-20b

$(-1)^k F(n)F(m-k) + (-1)^m L(k)F(n-m) + (-1)^n L(m)F(k-n) = 0$	FQ 11 (1973) B228 page 108
$(-1)^k L(n)F(m-k) + (-1)^m F(k)F(n-m) + (-1)^n F(m)F(k-n) = 0$	FQ 11 (1973) B229 page 108
$5 F(jk+r) F(ju+v) = L(j(k+u)+(r+v)) - (-1)^{ju+v} L(j(k-u)+(r-v))$	FQ 16 General Identities For Linear Fibonacci And Lucas Summations, R T Hansen, 121-128
$F(jk+r) L(ju+v) = F(j(k+u)+(r+v)) + (-1)^{ju+v} F(j(k-u)+(r-v))$	FQ 16 General Identities For Linear Fibonacci And Lucas Summations, R T Hansen, 121-128
$L(jk+r) L(ju+v) = L(j(k+u)+(r+v)) + (-1)^{ju+v} L(j(k-u)+(r-v))$	FQ 16 General Identities For Linear Fibonacci And Lucas Summations, R T Hansen, 121-128
$5F(a)F(b) - L(c)L(d) = (-1)^r(5F(a-r)F(b-r) - L(c-r)L(d-r))$ *a+b=c+d for any integers a,b,c,d,r*	Johnson
$F(a) L(b) - F(c) L(d) = (-1)^r(F(a-r) L(b-r) - F(c-r) L(d-r)$ *with a+b=c+d*	Johnson-32, special case of Johnson-44

Fibonacci and Lucas Factors

$\frac{F(kt)}{F(t)} = \sum_{i=0}^{(k-3)/2} (-1)^{it} L((k-2i-1)t) + (-1)^{(k-1)t/2}$ *for ODD* $k \ge 3$	Vajda-85
$\frac{F(kt)}{F(t)} = \sum_{i=0}^{k/2-1} (-1)^{it} L((k-2i-1)t)$ *for EVEN* $k \ge 2$	Vajda-86
$\frac{L(kt)}{L(t)} = \sum_{i=0}^{(k-3)/2} (-1)^{i(t+1)} L((k-2i-1)t) + (-1)^{(k-1)(t+1)/2}$ *for ODD* $k \ge 3$	Vajda-87
L(t) is not a factor of L(kt) for even k	
$\frac{F(kt)}{L(t)} = \sum_{i=0}^{k/2-1} (-1)^{i(t+1)} F((k-2i-1)t)$ *for EVEN* $k \ge 2$	Vajda-88
L(t) is not a factor of F(kt) for odd k	
p^k *divides* $F(np)$ *but* p^{k+1} *does not if p is odd and* $k \ge 1$	Lucas (1876)

Higher Order Fibonacci and Lucas

Fibonacci and Lucas cubed

$F(3n) = F(n + 1)^3 + F(n)^3 - F(n - 1)^3$	B&Q(2003)-Identity 232
$F(n + 1)F(n + 2)F(n + 6) - F(n + 3)^3 = (-1)^n F(n)$ $F(n)F(n + 4)F(n + 5) - F(n + 3)^3 = (-1)^{n+1} F(n + 6)$	FQ 41 (2003) pg 142, Melham. The second is a variant with -n for n and using Vajda-2

$F(n-2)F(n-1)F(n+3) - F(n)^3 = (-1)^{n-1}F(n-3)$ $F(n+2)F(n+1)F(n-3) - F(n)^3 = (-1)^nF(n+3)$	Fairgrieve and Gould (2005) versions of the above two formulae of Melham
$F(n-2)F(n+1)^2 - F(n)^3 = (-1)^{n-1} F(n-1)$ $F(n+2)F(n-1)^2 - F(n)^3 = (-1)^n F(n+1)$	Fairgrieve and Gould (2005)
$F(n+a+b)F(n-a)F(n-b) - F(n-a-b)F(n+a)F(n+b)$ $= (-1)^{n+a+b}F(a)F(b)(a+b)L(n)$	Melham (2011) Theorem 1
$F(n+a+b-c)F(n-a+c)F(n-b+c) - F(n-a-b+c)F(n+a)F(n+b)$ $= (-1)^{n+a+b+c}F(a+b-c)(F(c)F(n+a+b-c) + (-1)^cF(a-c)F(b-c)L(n))$	Melham (2011) Theorem 5
$F(i+j+k) =$ $F(i+1)F(j+1)F(k+1) + F(i)F(j)F(k) - F(i-1)F(j-1)F(k-1)$ *for any integers i,j,k*	Johnson's (6)
$L(5n) = L(n) (L(2n) + 5F(n) + 3)(L(2n) - 5F(n) + 3)$, *n odd*	**Aurifeuille's Identity (1879)** FQ 42 (2004) R S Melham, pgs 155-160

Fibonacci and Lucas to the fourth

$F(n-1)^2F(n+1)^2 - F(n-2)^2F(n+2)^2 = 4(-1)^nF(n)^2$	Melham (2011) 21
$F(n-3)F(n-1)F(n+1)F(n+3) - F(n)^4 = (-1)^nL(n)^2$	Melham (2011) 22
$F(n)^2 F(m + 1) F(m - 1) - F(m)^2 F(n + 1) F(n - 1)$ $= (-1)^{n-1} F(m + n) F(m - n)$	Vajda-32
$F(n - 2)F(n - 1)F(n + 1)F(n + 2) + 1 = F(n)^4$	**Gelin-Cesàro Identity** (1880) (see Dickson page 401) FQ 41 (2003) pg 142, B&Q(2003)-Identity 31
$F(n+a+b+c)F(n-a)F(n-b)F(n-c) - F(n-a-b-c)F(n+a)F(n+b)F(n+c)$ $= (-1)^{n+a+b+c}F(a+b)F(a+c)(b+c)F(2n)$	Melham (2011) Theorem 2
$F(n+a+b+c-d)F(n-a+d)F(n-b+d)F(n-c+d) - F(n-a-b-c+2d)F(n+a)F(n+b)F(n+c)$ $= (-1)^{n+a+b+c}F(a+b-d)F(a+c-d)F(b+c-d)F(2n+d)$	Melham (2011) Theorem 6
$L(n - 2)L(n - 1)L(n + 1)L(n + 2) + 25 = L(n)^4$	B&Q(2003)-Identity 56
$F(n)F(n+1)F(n+3)F(n+4) + 1 = F(n+2)^4$	Hoggatt-I29, Simson(1753)
$(F(n)^2 + F(n+1)^2 + F(n+2)^2)^2 = 2 (F(n)^4 + F(n+1)^4 + F(n+2)^4)$	**Candido's Identity (1951)** FQ 42 (2004) R S Melham, pgs 155-160
$[L(n-1)L(n+2)]^2 + [2L(n)L(n+1)]^2 = [5F(2n+1)]^2$	Wulczyn *FQ* 18 (1980) pg 188 special case of **Generalised Fibonacci Pythagorean Triples**

Fibonacci and Lucas Higher Powers

$F(n)F(n+1)F(n+2)F(n+4)F(n+5)F(n+6) + L(n+3)^2 =$ $[F(n+3)(2F(n+2)F(n+4) - F(n+3)^2)]^2$	J Morgado **Note on some results of A F Horadam and A G Shannon concerning Catalan's Identity on Fibonaci Numbers** *Portugaliae Math.* 44 (1987) pgs 243-252

$\left(\frac{L(n) + \sqrt{5}\, F(n)}{2}\right)^k = \frac{L(kn) + \sqrt{5}\, F(kn)}{2}$	**De Moivre Analogue**, S Fisk (1963) FQ 1.2 Problem B-10, pg 85. Hoggatt-I44
$\left(\frac{L(n) - \sqrt{5}\, F(n)}{2}\right)^k = \frac{L(kn) - \sqrt{5}\, F(kn)}{2}$	**De Moivre Analogue**

G Formulae

G(i) is the General Fibonacci series. It has the same recurrence relation as Fibonacci and Lucas, namely **G(n+2) = G(n+1) + G(n) for all integers n (i.e. n can be negative)**, but the "starting values" of G(0)=a and G(1)=b can be specified. It therefore includes both series them both as special cases. To make it clear which starting values for G(0)=a and G(1)=b are being used, we write G(a,b,i) for G(i). Hoggatt and others use the letter H for series G. For example:

- If G(0)=0 and G(1)=1 we have 0,1,1,2,3,5,8,13,.. the Fibonacci series, i.e. G(0,1,i) = F(i);
- G(0)=2 and G(1)=1 gives 2,1,3,4,7,11,18,.. the Lucas series, i.e. G(2,1,i) = L(i);

Basic G Formulae

Two independent G series are denoted G(n) and H(n).

$\sqrt{5}\, G(n) = (\, G(0)\, phi + G(1)\,)\, Phi^n + (G(0)\, Phi - G(1))\, (\, -phi\,)^n$	Vajda-55/56, Dunlap-77, B&Q(2003)-Identity 244
$G(n + 2) = G(n + 1) + G(n)$	Vajda-3, Dunlap-4
$G(n) = G(0)\, F(n - 1) + G(1)\, F(n)$	B&Q(2003)-Identity 37
$F(n) = \frac{G(0)\, G(n+1) - G(1)\, G(n)}{G(0)G(2) - G(1)^2}$	*Amer Math Montly* (2005) "Fibonacci, Chebyshev and Orthoganal Polynomials" D Aharonov, A Beardam, K Driver, p612-630
$2\, G(k) = (\, 2\, G(1) - G(0)\,)\, F(k) + G(0)\, L(k)$	Johnson-46
$G(-n) = (-1)^n\, (G(0)\, F(n + 1) - G(1)\, F(n))$	-
$G(n + m) = F(m - 1)\, G(n) + F(m)\, G(n + 1)$	Vajda-8, Dunlap-33, B&Q(2003)-Identity 38, Johnson-40
$G(n - m) = (-1)^m\, (F(m + 1)\, G(n) - F(m)\, G(n + 1))$	Vajda-9, Dunlap-34, B&Q(2003)-Identity 47
$G(n + m) + (-1)^m\, G(n - m) = L(m)\, G(n)$	Vajda-10a, Dunlap-35, B&Q(2003)-Identity 45, Bergum & Hoggatt (1975) (36) and (38)
$G(n + m) - (-1)^m\, G(n - m) = F(m)\, (\, G(n-1) + G(n+1))$	B&Q(2003)-Identity 48, , Bergum & Hoggatt (1975) (37) and (39)
$F(m)\, (G(n - 1) + G(n + 1)) = G(n + m) - (-1)^m\, G(n - m)$	Vajda-10b, Dunlap-36
$G(m)\, F(n) - G(n)\, F(m) = (-1)^{n+1}\, G(0)\, F(m - n)$	Vajda-21a
$G(m)\, F(n) - G(n)\, F(m) = (-1)^m\, G(0)\, F(n - m)$	Vajda-21b
$G(m+k)\, F(n+k) + (-1)^{k+1}\, G(m)\, F(n) = F(k)\, G(m + n + k)$	Howard(2003)

G Formulae of Order 2 or more

These formulae include terms which are a product of two G numbers either from the same G series of from two different G series i.e. with different index 0 and 1 values. Where the series may be different they are denoted G and H e.g. special cases include G = F (i.e. Fibonacci) and H = L (i.e. Lucas), or they could also be the same series G=H.

$G(n+i)\,H(n+k) - G(n)\,H(n+i+k) = (-1)^n\,(G(i)\,H(k) - G(0)\,H(i+k))$	Vajda-18 (corrected), B&Q(2003)-Identity 44 a special case of Johnson's:
$G(p)H(q) - G(r)H(s) = (-1)^n[\,G(p-n)H(q-n) - G(r-n)H(s-n)\,]$ *if p+q = r+s and p,q,r,s,n are integers*	Johnson-44
$G(n+1)\,G(n-1) - G(n)^2 = (-1)^n\,(G(1)^2 - G(0)\,G(2))$	Vajda-28, B&Q(2003)-Identity 46
$4\,G(n-1)G(n) + G(n-2)^2 = G(n+1)^2$	B&Q(2003)-Identity 65
$G(n+3)^2 + G(n)^2 = 2(\,G(n+1)^2 + G(n+2)^2\,)$	B&Q(2003)-Identity 70
$G(i+j+k) = F(i+1)F(j+1)G(k+1) + F(i)F(j)G(k) - F(i-1)F(j-1)G(k-1)$ *for any integers i,j,k*	Johnson's (39a)
$4G(i)^2G(i+1)^2 + G(i-1)^2G(i+2)^2 = (\,G(i)^2 + G(i+1)^2\,)^2$	Generalised Fibonacci Pythagorean Triples A F Horadam **Special Properties of the Sequence w_n(a,b;p,q)** *FQ* 5 (1967) pgs 424-434
$G(n+2)G(n+1)G(n-1)G(n-2) + (\,G(2)G(0) - G(1)^2\,)^2 = G(n)^4$	B&Q(2003)-Identity 59

Summations

This section has formulae that sum a variable number of terms.

Fibonacci and Lucas Summations

These formulae involve a sum of Fibonacci or Lucas numbers only.

$\sum_{i=0}^{n} F(i) = F(n+2) - 1$	Hoggatt-I1, Lucas(1878), B&Q 2003-Identity 1
$\sum_{i=0}^{n} (-1)^i\,F(i) = (-1)^n\,F(n-1) - 1$	B&Q 2003-Identity 21
$\sum_{i=0}^{n} L(i) = L(n+2) - 1$	Hoggatt-I2
$\sum_{i=a}^{n} F(i) = F(n+2) - F(a+1)$	-
$\sum_{i=a}^{n} L(i) = L(n+2) - L(a+1)$	-

$\sum_{i=0}^{n} F(2i) = F(2n+1) - 1,\ n \ge 0$	Hoggatt-I6, Lucas(1878), B&Q(2003)-Identity 12
$\sum_{i=1}^{n} F(2i-1) = F(2n),\ n \ge 1$	Hoggatt-I5, Lucas(1878), B&Q(2003)-Identity 2
$\sum_{i=1}^{n} L(2i-1) = L(2n) - 2$	-
$\sum_{i=1}^{n} 2^{n-i} F(i-1) = 2^n - F(n+2)$	Vajda-37a(adapted), Dunlap-42(adapted), B&Q(2003)-Identity 10
$\sum_{i=0}^{n} 2^i L(i) = 2^{n+1} F(n+1)$	B&Q(2003)-Identity 236
$\sum_{i=0}^{n} F(3i-1) = \frac{F(3n+1)+1}{2}$	B&Q(2003)-Identity 24
$\sum_{i=0}^{n} F(3i) = \frac{F(3n+2)-1}{2}$	B&Q(2003)-Identity 25
$\sum_{i=0}^{n} F(3i+1) = \frac{F(3n+3)}{2}$	B&Q(2003)-Identity 23
$\sum_{i=0}^{n} F(4i) = F(2n+1)^2 - 1$	B&Q 2003-Identity 27
$\sum_{i=0}^{n} F(4i+1) = F(2n+1)F(2n+2)$	B&Q 2003-Identity 26
$\sum_{i=0}^{n} F(4i+2) = F(2n+1)F(2n+3) - 1$	B&Q 2003-Identity 29
$\sum_{i=0}^{n} F(4i+3) = F(2n+3)F(2n+2)$	B&Q 2003-Identity 28
$\sum_{i=0}^{n} (-1)^i L(n-2i) = 2F(n+1)$	Vajda-97, Dunlap-54
$\sum_{i=0}^{n} (-1)^i L(2n-2i+1) = F(2n+2)$	B&Q(2003)-Identity 55

Decimal (and other bases) fractions

We saw in The Fibonacci Series as a Decimal Fraction that the Fibonacci series occurs naturally as the decimal expansion of a simple fraction in several ways:

$$1/89 = 0.0\ 1\ 1\ 2\ 3\ 5\ \ldots$$

$$1/9899 = 0.00\ 01\ 01\ 02\ 03\ 05\ 08\ 13\ 21\ \ldots$$

with a varying number of decimal digits before the Fibonacci numbers overlap and the series is obscured. This section gives formulae for these fractions for various subsequences of Fibonacci and General Fibonacci series.

$$\sum_{k=1}^{\infty} 10^{-n(k+1)}F(ak) = \frac{F(a)}{10^{2n} - 10^{n}L(a) - (-1)^{a}}$$ Hudson and Winans (1981)

If P(n) = a P(n-1) + b P(n-2) for n≥2; P(0) = c; P(1) = d and m and N are defined by $B^2 = m + Ba + b$, $N = cm + dB + bc$, *then*

$$\frac{N}{Bm} = \sum_{i=1}^{\infty} \frac{P(i-1)}{B^{i}}$$ Long (1981)

provided that abs($(a+\sqrt{(a^2+4b)})/(2B)$ *)<1 and* $| (a-\sqrt{(a^2+4b)})/(2B) | < 1$

Summations with fractions

$$\sum_{i=0}^{\infty} \frac{F(i)}{2^{i}} = 2$$ Vajda-60, Dunlap-51

$$\sum_{i=0}^{\infty} \frac{L(i)}{2^{i}} = 6$$ -

$$\sum_{i=0}^{\infty} \frac{F(i)}{r^{i}} = \frac{r}{r^2 - r - 1}$$ -

$$\sum_{i=0}^{\infty} \frac{L(i)}{r^{i}} = 2 + \frac{r+2}{r^2 - r - 1}$$ -

$$\sum_{i=1}^{\infty} \frac{i\,F(i)}{2^{i}} = 10$$ Vajda-61, Dunlap-52

$$\sum_{i=1}^{\infty} \frac{i\,L(i)}{2^{i}} = 22$$ -

$$\sum_{i=1}^{\infty} \frac{1}{F(2^{i})} = 4 - Phi = 3 - phi$$ Vajda-77(corrected), Dunlap-53(corrected)

$$\sum_{i=1}^{n} \frac{(-1)^{2^{i-1}r}}{F(2ir)} = \frac{(-1)^{r}\,F(\,r(2^{n}-1)\,)}{F(r)\,F(2^{n}\,r)}$$ Vajda-89

$$\sum \frac{1}{F(k-1)F(k+1)} = 1$$ R L Graham (1963) FQ 1.1, Problem B-9, pg 85, FQ 1.4 page 79

$k \ge 2$

$$\sum_{k \ge 2} \frac{F(k)}{F(k-1)F(k+1)} = 2$$ R L Graham (1963) FQ 1.1, Problem B-9, pg 85

$$\sum_{k \ge 2} \frac{(-1)^k}{F(k)F(k-1)} = Phi - 1$$ Johnson-11, Vajda-102

$$\sum_{k \ge 2} \frac{1}{F(2k+1)F(2k-1)} = Phi - 1$$ alternative form of Johnson-11

Order 2 summations

$\sum_{i=1}^{n} F(i)^2 = F(n)\, F(n+1)$	Vajda-45, Dunlap-5, Hoggatt-I3, Lucas(1878), Koshy-77, B&Q(2003)-Identity 9 (Identity 233 variant)
$\sum_{i=1}^{n} L(i)^2 = L(n)\, L(n+1) - 2$	Hoggatt-I4
$\sum_{i=1}^{2n-1} L(i)^2 = 5\, F(2n)\, F(2n-1)$	-
$\sum_{i=1}^{2n} F(i)\, F(i-1) = F(2n)^2$	Vajda-40, Dunlap-45
$\sum_{i=1}^{2n} L(i)\, L(i-1) = L(2n)^2 - 4$	-
$\sum_{i=1}^{2n+1} F(i)\, F(i-1) = F(2n+1)^2 - 1$	Vajda-42, Dunlap-47
$\sum_{i=1}^{2n+1} L(i)\, L(i-1) = L(2n+1)^2 - 5$	-
$5 \sum_{k=0}^{n} (-1)^{r(1+k)}\, F(r(1+k))^2 = (-1)^{r(n+1)} \frac{F((2n+3)r)}{F(r)} - 2n - 3$	Vajda-93
$\sum_{k=0}^{n} (-1)^{r(1+k)}\, L(r(1+k))^2 = (-1)^{r(n+1)} \frac{F((2n+3)r)}{F(r)} + 2n + 1$	Vajda-94
$\sum_{i=0}^{n-1} F(2i+1)^2 = \frac{F(4n) + 2n}{5}$	Vajda-95, B&Q(2003)-Identity 234
$\sum_{i=0}^{n} F(2i)^2 = \frac{F(4n+2) - 2n - 1}{5}$	Vajda page 70

$$\sum_{i=0}^{n-1} L(2i+1)^2 = F(4n) - 2n$$ Vajda-96, B&Q(2003)-Identity 54

$$\sum_{i=1}^{n} L(2i)^2 = F(4n+2) + 2n - 1$$ Vajda page 70

$$5\sum_{i=0}^{n} F(i)\,F(n-i) \begin{cases} = (n+1)\,L(n) - 2\,F(n+1) \\ = n\,L(n) - F(n) \end{cases}$$ Vajda-98, Dunlap-55, B&Q(2003)-Identity 58

$$\sum_{i=0}^{n} L(i)\,L(n-i) \begin{cases} = (n+1)\,L(n) + 2\,F(n+1) \\ = (n+2)\,L(n) + F(n) \end{cases}$$ Vajda-99, Dunlap-56, B&Q(2003)-Identity 57

$$\sum_{i=0}^{n} F(i)\,L(n-i) = (n+1)\,F(n)$$ Vajda-100, Dunlap-57, B&Q(2003)-Identity 35

$$\sum_{k=1}^{2n-1} (2n-k)\,F(k)^2 = F(2n)^2$$ V Hoggatt (1965) Problem B-53 FQ 3, pg 157

Summations of order > 2

$$10\sum_{i=1}^{n} F(i)^3 = F(3n+2) + 6(-1)^{n+1}F(n-1) + 5$$ *adapted from* Benjamin, Carnes, Cloitre (2009)

$$25\sum_{i=1}^{n} F(i)^4 = F(4n+2) + 4(-1)^{n+1}F(2n+1) + 6n + 3$$ see [A005969]

$$4\sum_{k=1}^{n} F(k)^6 = F(n)^5F(n+3) + F(2n)$$ Ohtsuka and Nakamura (2010) Theorem 1

$$4\sum_{k=1}^{n} L(k)^6 = L(n)^5L(n+3) + 125\,F(2n) - 128$$ Ohtsuka and Nakamura (2010) Theorem 2

$$F(m\,q) = F(m)\sum_{j=1}^{q} F(m-1)^{j-1}F(\,m(q-j)+1\,)$$ B&Q(2003)-Theorem 2

G Summations

Two independent G series are denoted G(n) and H(n).

$$\sum_{i=1}^{n} G(i) = G(n+2) - G(2)$$ L G Brökling (1964) FQ 2.1 Problem B-20 solution, pg76; Vajda-33; Dunlap-38; B&Q(2003)-Identity 39

$$\sum^{n} G(i) = G(n+2) - G(a+1)$$ -

$i = a$

$$\sum_{i=1}^{n} G(2i-1) = G(2n) - G(0)$$

Vajda-34, Dunlap-37, B&Q(2003)-Identity 61

$$\sum_{i=1}^{n} G(2i) = G(2n+1) - G(1)$$

Vajda-35, Dunlap-39, B&Q(2003)-Identity 62

$$\sum_{i=1}^{n} G(2i) - \sum_{i=1}^{n} G(2i-1) = G(2n-1) + G(0) - G(1)$$

Vajda-36, Dunlap-40

$$\sum_{i=1}^{n} 2^{n-i}\, G(i-1) \begin{array}{l} = 2^{n-1}(\ G(0) + G(3)\ ) - G(n+2) \\ = 2^{n}\ (\ G(0) + G(1)\ ) - G(n+2) \end{array}$$

Vajda-37, Dunlap-41,
B&Q(2003)-Identity 69

$$\sum_{i=1}^{4n+2} G(i) = L(2n+1)\, G(2n+3)$$

Vajda-38, Dunlap-43, B&Q(2003)-Identity 49

$$\sum_{i=1}^{2n} G(i)\, G(i-1) = G(2n)^2 - G(0)^2$$

Vajda-39, Dunlap-44, B&Q(2003)-Identity 41

$$\sum_{i=1}^{2n+1} G(i)\, G(i-1) = G(2\ n+1)^2 - G(0)^2 - G(1)^2 + G(0)G(2)$$

Vajda-41, Dunlap-46

$$\sum_{i=1}^{n} G(i+2)\, G(i-1) = G(n+1)^2 - G(1)^2$$

Vajda-43, Dunlap-48, B&Q(2003)-Identity 64

$$\sum_{k=0}^{n} G(m+kr) = \frac{1}{1+(-1)^r - L(r)} [\ G(m) - G(m+(n+1)r) + (-1)^r(G(m+nr) - G(m-r))\ ]$$

Fibonacci with a Golden Ring
Kung-Wei Yang *Mathematics Magazine* 70 (1997),
pp. 131-135.

$$\sum_{i=1}^{n} G(i)^2 = G(n)\, G(n+1) - G(0)\, G(1)$$

Vajda-44, Dunlap-49, B&Q(2003)-Identity 67

$$\sum_{i=0}^{\infty} \frac{G(a,b,i)}{r^i} = a + \frac{a+b\,r}{r^2 - r - 1}$$

Stan Rabinowitz,
"Second-Order Linear Recurrences" card,
Generating Function
special case (x=1/r, P=1, Q=-1)

$$\sum_{i=0}^{\infty} \frac{i\, G(a,b,i)}{r^i} = \frac{r\,(b\,r^2 - 2\,a\,r + b - a)}{(r^2 - r - 1)^2}$$

-

$$\sum_{i=1}^{2n-1} G(\ i\ )\, H(\ i\ ) = G\ (\ 2n\ )\, H(\ 2n-1) - G(0)\, H(1)$$

B&Q(2003)-Identity 42

Summations with Binomial Coefficients

$$\sum_{i=1}^{n} \binom{n-i}{i-1} = F(n)$$

B&Q(2003) Identity-4

∞

$\sum_{i=0} \binom{n-i-1}{i} = F(n)$	Vajda-54(corrected), Dunlap-84(corrected)
$\sum_{i=0}^{n} \binom{n+i}{2i} = F(2n+1)$	B&Q(2003)-Identity 165
$\sum_{i=0}^{n-1} \binom{n+i}{2i+1} = F(2n)$	B&Q(2003)-Identity 166
$\sum_{k=0}^{n} \binom{n}{k} F(k) = F(2n)$	S Basin & V Ivanoff (1963) Problem B-4, FQ 1.1 pg 74, FQ1.2 pg 79; B&Q(2003)-Identity 6
$\sum_{k=0}^{n} \binom{n}{k} (-1)^{k+1} F(k) = F(n)$	I Ruggles (1963) FQ 1.2 pg 77
$\sum_{k=0}^{n} \binom{n}{k} (-1)^{k} L(k) = L(n)$	I Ruggles (1963) FQ 1.2 pg 77
$\sum_{k=0}^{n} \binom{n}{k} F(p-k) = F(p+n)$	B&Q(2003)-Identity 20
$\sum_{k=1}^{n} \binom{n}{k} 2^{k} F(k) = F(3n)$	B&Q(2003)-Identity 238, Vajda-68
$\sum_{i=0}^{n} \binom{n+1}{i+1} F(i) = F(2n+1) - 1$	Vajda-50, Dunlap-82
$\sum_{i=0}^{2n} \binom{2n}{i} F(2i+p) = 5^{n} F(2n+p)$	Hoggatt-I41 (special case p=0: Vajda-69, Dunlap-85)
$\sum_{i=0}^{2n} \binom{2n}{i} L(2i) = 5^{n} L(2n)$	Vajda-71, Dunlap-87
$\sum_{i=0}^{2n+1} \binom{2n+1}{i} F(2i+p) = 5^{n} L(2n+1+p)$	Hoggatt-I42 (special case p=0: Vajda-70, Dunlap-86)
$\sum_{i=0}^{2n+1} \binom{2n+1}{i} L(2i) = 5^{n+1} F(2n+1)$	Vajda-72, Dunlap-88
$\sum^{2n} \binom{2n}{i} F(i)^{2} = 5^{n-1} L(2n)$	Vajda-73, Dunlap-89,Hoggatt-I45

$i=0$

$\sum_{i=0}^{2n} \binom{2n}{i} L(i)^2 = 5^n L(2n)$	Vajda-75, Dunlap-91, Hoggatt-I46
$\sum_{i=0}^{2n+1} \binom{2n+1}{i} F(i)^2 = 5^n F(2n+1)$	Vajda-74, Dunlap-90, Hoggatt-I47
$\sum_{i=0}^{2n+1} \binom{2n+1}{i} L(i)^2 = 5^{n+1} F(2n+1)$	Vajda-76, Dunlap-92
$\sum_{i=0}^{\infty} 5^i \binom{n}{2i+1} = 2^{n-1} F(n)$	Vajda-91, B&Q(2003)-Identity 235, Catalan 1857
$\sum_{i=0}^{\infty} 5^i \binom{n}{2i} = 2^{n-1} L(n)$	Vajda-92, B&Q(2003)-Identity 237, Catalan (1857)-see Vajda pg 69
$\sum_{i=0}^{k} \binom{k}{i} F(n)^i F(n-1)^{k-i} F(i) = F(kn)$	Rabinowitz-17 (special case of Vajda-66)
$\sum_{i=0}^{k} \binom{k}{i} F(n)^i F(n-1)^{k-i} L(i) = L(kn)$	Rabinowitz-17 (special case of Vajda-66)
$\sum_{i=0}^{p} \binom{p}{i} F(t)^i F(t-1)^{p-i} G(m+i) = G(m+tp)$	Vajda-66
$\sum_{i\ge 0} \sum_{j\ge 0} \binom{n-i}{j} \binom{n-j}{i} = F(2n+3)$	B&Q(2003) Identity 5
$\frac{F(rn)}{F(r)} = \sum_{k=0}^{[(n-1)/2]} (-1)^{k(r-1)} \binom{n-k-1}{k} L(r)^{n-1-2k}$	Lucas (1878) equns 74-76, this form due to Hoggatt and Lindt (1969), see Gould (1977)

Powers of Fibonacci and Lucas as Sums

$5^{k/2} F(t)^k = \sum_{i=0}^{(k-1)/2} \binom{k}{i} (-1)^{i(t+1)} \sqrt{5}\, F((k-2i)t)$, k odd	Vajda-80
$5^{k/2} F(t)^k = \sum_{i=0}^{k/2-1} \binom{k}{i} (-1)^{i(t+1)} L((k-2i)t) + \binom{k}{k/2} (-1)^{(t+1)k/2}$, k even	Vajda-81
$L(t)^k = \sum_{i=0}^{(k-1)/2} \binom{k}{i} (-1)^{it} L((k-2i)t)$, k odd	Vajda-78

$L(t)^k = \sum_{i=0}^{k/2-1} \binom{k}{i} (-1)^{it} L((k-2i)t) + \binom{k}{k/2} (-1)^{tk/2}$, k even — Vajda-79

$F_m^k F_n = (-1)^{kr} \sum_{h=0}^{k} \binom{k}{h} (-1)^h F_r^h F_{r+m}^{k-h} F_{n+kr+hm}$ — On a General Fibonacci Identity, J H Halton, Fib Q, 3 (1965), pp 31-43

Summations with Binomials and G Series

$\sum_{i=0}^{n} \binom{n}{i} G(i) = G(2n)$ — I Ruggles (1963) FQ 1.2 pg 77; Vajda-47; Dunlap-80

$\sum_{i=0}^{n} \binom{n}{i} 2^i G(i) = G(3n)$ — B&Q(2003)-Identity 239

$\sum_{i=0}^{n} \binom{n}{i} G(p-i) = G(p+n)$ — Vajda-46, Dunlap-79, B&Q(2003)-Identity 40

$\sum_{i=0}^{n} \binom{n}{i} G(p+i) = G(p+2n)$ — Vajda-49, Dunlap-81

$\sum_{i=0}^{p} (-1)^{p-i} \binom{p}{i} G(n+i) = G(n-p)$ — Vajda-51, Dunlap-83

Trigonometric Formulae

$F(n) = \prod_{k=1}^{floor((n-1)/2)} \left(3 + 2\cos\frac{2k\pi}{n}\right), n \ge 1$ — See **Recursive Properties of Trigonometric Products** R J Hendel & C K Cook in *Applications of Fibonacci Numbers* Vol 6 (1996) pgs 201-214

$L(n) = \prod_{k=0}^{floor((n-2)/2)} \left(3 + 2\cos\frac{(2k+1)\pi}{n}\right), n \ge 2$ — See **Recursive Properties of Trigonometric Products** R J Hendel & C K Cook in *Applications of Fibonacci Numbers* Vol 6 (1996) pgs 201-214

Hyperbolic Functions

Here we use g for ln(Phi), the natural log of Phi. cosh(g)=√5 / 2. There are several derivations of formulae above using hyperbolic functions in chapter XI of Vajda.

$F(2n) = \frac{2}{\sqrt{5}} \sinh(2ng)$ — from Binet's formula

$F(2n) = \frac{2}{\sqrt{5}} sinh(2ng)$ from Binet's formula

$= \frac{sinh(2ng)}{cosh(g)}$

$F(2n+1) = \frac{2}{\sqrt{5}} cosh((2n+1)g)$ from Binet's formula

$= \frac{cosh((2n+1)g)}{cosh(g)}$

$L(2n) = 2\ cosh(ng)$ from Binet's formula

$L(2n+1) = 2\ sinh(ng)$ from Binet's formula

Complex Numbers

$F(n) = \frac{2\ i^{1-n}}{\sqrt{5}} sin(-i\ n\ ln(i\ Phi))$	from Rabinowitz-7 corrected
$F(n) = \frac{2\ i^{-n}}{\sqrt{5}} sinh(n\ ln(i\ Phi))$	from Rabinowitz-7 corrected
$L(n) = 2\ i^{-n}\ cos(-i\ n\ ln(i\ Phi))$	from Rabinowitz-7 corrected
$L(n) = 2\ i^{-n}\ cosh(n\ ln(i\ Phi))$	from Rabinowitz-7 corrected
$\sqrt{1+2i} = \sqrt{Phi+i}/\sqrt{Phi} = [1+i, \overline{2+2i}]$	I J Good (1993)
$\sqrt{1+i/2} = \sqrt{Phi+i}/\sqrt{2} + i\sqrt{(phi-1)}/\sqrt{2} = [(1+i)/2, \overline{1+i}]$	I J Good (1993)

Generating Functions

Ordinary Generating Functions

For many series, $S(n)$, we can find a (simple) power-series expression in x (that is, a sum of powers of x) where the coefficients of the nth power of x is the nth term in the series, $S(n)$:

$$G(x) = \sum_{i=0}^{\infty} S(i)\ x^i = S(0) + S(1)\ x + S(2)\ x^2 + S(3)\ x^3 + ...$$

Such an expression, $G(x)$, if it exists for the series S is called the *generating function for* S or GF for short.

To shift to the right (insert a 0 at the start of the series so all other terms have an index increased by 1), multiply the GF by x; to shift to the left, divide by x.
There is much more on GFs on my Fibonomials page.

Fibonacci(n) 0,1,1,2,3,...	$\frac{x}{1-x-x^2}$	Lucas(n) 2,1,3,4,7,...	$\frac{2-x}{1-x-x^2}$	G(a,b,n) a,b,a+b,a+2b,...	$\frac{a+(b-a)x}{1-x-x^2}$
Fibonacci(2n) 0,1,3,8,21,...	$\frac{x}{x^2-3x+1}$	Lucas(2n) 2,3,7,18,...	$\frac{2-3x}{x^2-3x+1}$	G(a,b,2n) a,a+b,2a+3b,...	$\frac{a+(b-2a)x}{x^2-3x+1}$
Fibonacci(2n+1) 1,2,5,13,...	$\frac{1-x}{x^2-3x+1}$	Lucas(2n+1) 1,4,11,29,...	$\frac{1+x}{x^2-3x+1}$	G(a,b,2n+1) b,a+2b,3a+5b,...	$\frac{b+(a-b)x}{x^2-3x+1}$
Fibonacci(3n) 2,8,34,144,...	$\frac{2x}{1-4x-x^2}$	Lucas(3n) 2,4,18,76,...	$\frac{2-4x}{1-4x-x^2}$	G(a,b,3n) a,a+2b,5a+8b,...	$\frac{a+(2b-3a)x}{1-4x-x^2}$
Fibonacci(3n+1)	$1-x$	Lucas(3n+1)	$3x+1$	G(a,b,3n+1)	$b+(2a-b)x$

Fibonacci(3n+1) 1,3,13,55,...	$\frac{[illegible]}{1-4x-x^2}$	Lucas(3n+1) 3,11,47,199,...	$\frac{[illegible]}{1-4x-x^2}$	G(a,b,3n+1) a+b,3a+5b,13a+21b,...	$\frac{[illegible]}{1-4x-x^2}$
Fibonacci(3n+2) 1,5,21,89,...	$\frac{x+1}{1-4x-x^2}$	Lucas(3n+2) 2,4,18,76,...	$\frac{3-x}{1-4x-x^2}$	G(a,b,3n+2) a,a+2b,5a+8b,...	$\frac{a+b+(b-a)x}{1-4x-x^2}$
Fibonacci(k n)	$\frac{F(k)x}{1-L(k)x+(-1)^k x^2}$	Lucas(k n)	$\frac{2-L(k)x}{1-L(k)x+(-1)^k x^2}$	G(a,b,kn)	$\frac{a+(F(k)b-F(k+1)a)x}{1-L(k)x+(-1)^k x^2}$
Fibonacci(n)^2 $0^2,1^2,1^2,2^2,3^2,...$	$\frac{x-x^2}{1-2x-2x^2+x^3}$	Lucas(n)^2 $2^2,1^2,3^2,4^2,...$	$\frac{4-7x-x^2}{1-2x-2x^2+x^3}$	G(a,b,n)^2 $a^2,b^2,(a+b)^2,...$	$\frac{a^2+(b^2-2a^2)x-(a-b)^2x^2}{1-2x-2x^2+x^3}$
Fib(n)Fib(n+1) 1×1,1×2,2×3,3×5,...	$\frac{x}{1-4x-x^2}$	Lucas(n)Lucas(n+1) 2×1,1×3,3×4,4×7,...	$\frac{2-x+2x^2}{1-2x-2x^2+x^3}$	G(a,b,n)G(a,b,n+1) ab,b(a+b),(a+b)(a+2b),...	$\frac{ab+b(b-a)x+a(a-b)x^2}{1-2x-2x^2+x^3}$
Fibonacci(n)^3 $0^3,1^3,1^3,2^3,3^3,...$	$\frac{x^3+2x^2-x}{1-3x-6x^2+3x^3+x^4}$	Lucas(n)^3 $2^3,1^3,3^3,4^3,...$	$\frac{8-23x-24x^2+x^3}{1-3x-6x^2+3x^3+x^4}$		

Replacing x by x^2 in a GF inserts 0's between all values of the original series.
The series of even-indexed Fibonacci numbers only is the series 0,1,1,2,3,5,8,...
so it has the same GF as Fibonacci(2n) but with x^2 replacing x: $x^2/(x^4 - 3x^2 + 1)$ for the series 0,0,1,0,3,0,8,0,21,... .

The GF of 1,2,5,13,... is that of Fib[2n+1] which is $(1 - x)/(x^2 - 3x - 1)$
so 1,0,2,0,5,0,13,... has GF $(1 - x^2)/(x^4 - 3x^2 - 1)$
To insert an extra 0 at the start, multiply the GF by x: $x(1 - x^2)/(x^4 - 3x^2 - 1)$
So the GF for the odd-indexed Fibonacci numbers only *in their correct positions in the Fibonacci series so that Fib[2n+1] is the coefficient of* x^{2n+1} is therefore $x(1-x^2)/(x^4 - 3x^2 + 1)$ for the series 0,1,0,2,0,5,0,13,... .

Adding these two series and GFs, that is, the Fib[2n] as the coefficient of x^{2n} and Fib[2n+1] as the coefficient of x^{2n+1} should then give the complete Fibonacci series:

0,0,1,0,3,0,8, 0,21, ... +
0,1,0,2,0,5,0,13, 0, ...
0,1,1,2,3,5,8,13,21,...

We can check that $x^2/(x^4 - 3x^2 + 1) + x(1-x^2)/(x^4 - 3x^2 + 1) = x/(1 - x - x^2)$
which is the GF of 0,1,1,2,3,5,8,13,21,... as required!

Multiplying a GF by a constant k multiples all the members of the series by k.
A series formed by adding two series S(n) and T(n) element-wise to form the series S(n)+T(n), has a GF which is the sum of the two separate GFs.
Check that *a Fib[n-1] + b Fib[n]* gives the GF of G(a,b).

Exponential Generating Functions

$$\sum_{n}^{\infty} F(n)\frac{z^n}{n!} = \frac{e^{Phi\,z} - e^{-phi\,z}}{\sqrt{5}},\ z \text{ in } C$$

See, e.g., **Solving Linear Recurrences from Differential Equations in the Exponential Manner and Vice Versa** W Oberschelp in *Applications of Fibonacci Numbers* Vol 6 (1996) pages 365-380

References

(*) above indicates a private communication.

: *a book;*
: *an article (chapter, paper) in a book (journal);*
: *a web resource.*
FQ : *The Fibonacci Quarterly*

Arranged in alphabetical order of author:

A T Benjamin, J J Quinn Proofs That Really Count Mathematical Association of America, 2003, ISBN 0-88385-333-7, hardback, 194 pages. shown as B&Q(2003) in the Table above

Art Benjamin and Jennifer Quinn have a wonderful knack of presenting proofs that involve counting arrangements of dominoes and tiling patterns in two ways that convince you that a formula really *is* true and not just "proved"! The identities proved mainly involve Fibonacci, Lucas and the General Fibonacci series with chapters on continued fractions, binomial identites, the Harmonic and Stirling numbers too. There is so much here to inspire students to find proofs for themselves and to show that proofs can be fun too! **Important notation difference:** Benjamin and Quinn use f_n for the Fibonacci number F(n+1)

Bergum and Hoggatt (1975)
G. E. Bergum and V. E. Hoggatt, Jr. Sums and Products for Recurring Sequences, *Fib Q* 13 (1975), pages 115-120.

Benjamin, Carnes, Cloitre (2009)
Recounting the Sums of Cubes of Fibonacci Numbers A T Benjamin, T A. Carnes, B Cloitre, *Congressus Numerantium, Proceedings of the Eleventh International Conference on Fibonacci Numbers and their Applications*, William Webb (ed.), Vol 194, pp. 45-51, 2009.

L E Dickson History of the Theory of Numbers: Vol 1 Divisibility and Primality
is a classic and monumental reference work on all aspects of Number Theory in 3 volumes (volume II is on Diophantine Analysis and volume III on Quadratic and Higher Forms). Although not up-to-date (the original edition was 1952) it is still a comprehensive summary of useful historical and early references on all aspects of Number Theory. The link is to a new cheap Dover paperback edition (2005) of Volume 1 which contains the most about Fibonacci Numbers, Lucas numbers and the golden section: see Chapter XV11 on **Recurring Series, Lucas' u_n, v_n** where he uses *the series of Pisano* for what we now call the *Fibonacci numbers*.

R A Dunlap, The Golden Ratio and Fibonacci Numbers World Scientific Press, 1997, 162 pages.

An introductory book strong on the geometry and natural aspects of the golden section but it does not include much on the mathematical detail. Beware - some of the formulae in the Appendix are wrong! Dunlap has copied them from Vajda's book (see below) and he has faithfully preserved all of the original errors! The formulae on this page (that you are now reading) are corrected versions and have been verified.

Fairgrieve and Gould (2005)
Product Difference Fibonacci Identities of Simson, Gelin-Cesáro, Tagiuri and Generalizations. S Fairgrieve and H W Gould, *The Fibonacci Quarterly* 43 (2005), 137-141.

Gould (1977)
H W Gould, A Fibonacci Formula of Lucas and its Subsequent Manifestations and Rediscoveries , *Fibonacci Quarterly* vol 15 (1977) pages 25-29

R L Graham, D E Knuth, O Patashnik Concrete Mathematics Second Edition (1994), hardback, Addison-Wesley.

No - this is not a book about proportions of sand to cement when laying foundations for buildings. The title is meant as an antidote to the "Abstract Mathematics" courses so often found in the curricullum of a university maths degree.
As such, it is ***the*** book to dip into if you want to go really deeply into any part of the mathematics covered on this Fibonacci and Phi site. However, it quickly gets to an advanced mathematics undergraduate level after some nice introductions to every topic. There are notes left in the margins which were inserted by students taking the original courses based on this book at Stanford university and they are interesting, often useful and sometimes quite funny.

V E Hoggatt Jr "Fibonacci and Lucas Numbers" published by The Fibonacci Association, 1969 (Houghton Mifflin).
A very good introduction to the Fibonacci and Lucas Numbers written by a founder of the Fibonacci Quarterly.

Hoggatt and Lind (1969)
V E Hoggatt Jr, D A Lind, Compositions and Fibonacci Numbers, *The Fibonacci Quarterly*, Vol. 7, No. 3 (Oct., 1969), pp. 253-266.

F T Howard (2003) "The Sum of the Squares of Two Generalized Fibonacci Numbers" *FQ* vol 41 pages 80-84.

Hudson and Winans (1981)
A Complete Characterization of the Decimal Fractions That Can Be Represented as $\Sigma\ 10^{k(a+1)}F_{ai}$, where F_{ai} is the ai^{th} Fibonacci Number R H Hudson, C F Winans *The Fibonacci Quarterly* 19, no. 5 (1981) pages 414-421.
See also:
A Complete Characterization Of B-Power Fractions That Can Be Represented As Series Of General N-Bonacci Numbers J-Z Lee, J-S Lee *Fibonacci Quarterly* 25 (1987) pages 72-75.

I J Good Complex Fibonacci And Lucas Numbers, Continued Fractions, And The Square Root Of The Golden Ratio, *Fib Q* 31 (1993) pages 7-19

R Johnson (Durham university) has an excellent web page on the power of matrix methods to establish many Fibonacci formula with ease (but it does rely on at least undergraduate level matrix mathematics). See the **Matrix methods for Fibonacci and Related Sequences** link to a Postscript and PDF version on his Fibonacci Resources web page.
The latest version (Nov 12, 2004) contains an appendix showing how formulae developed in Johnson's paper can prove almost all the identities here in my table above.

D E Knuth The Art of Computer Programming: Vol 1 Fundamental Algorithms hardback, Addison-Wesley third edition (1997).
The paperback is now out of print and hard to find. This is the first of three volumes and an absolute must for all computer scientist/mathematicians.

T Koshy Fibonacci and Lucas Numbers with Applications, Wiley-Interscience, 2001, 648 pages.
This is a new book packed full of an amazing number of Fibonacci and related equations, culled from the pages of the *Fibonacci Quarterly*. Although initially impressive in its size and breadth, be aware that there are far too many typos, errors and missing or irrelevant conditions in many of its formulae as well as some glaring omissions and misattributions particularly with respect to the original references for a number of the formulae. Although Fibonacci representations of integers are included Zeckendorf himself is never even mentioned! There are lots of exercises with answers to the odd-numbered questions.

Long (1981)
The Decimal Expansion Of 1/89 And Related Results C Long *Fibonacci Quarterly* 19 (1985) pages 53-55

E Lucas, "Théorie des Fonctions Numériques Simplement Périodiques" in *American Journal of Mathematics* vol 1 (1878) pages 184-240 and 289-321

Journal of Mathematics vol 1 (1878) pages 184-240 and 289-321.

Reprinted as The Theory of Simply Periodic Functions, the Fibonacci Association, 1969.

R S Melham (1999) "Families of Identities Involving Sums of Powers of the Fibonacci and Lucas Numbers" *FQ* vol 37, pages 315-319.

R S Melham (2011), On Product Difference Fibonacci Identities Article A10, Integers, vol 11

Ohtsuka and Nakamura (2010)
A New Formula For The Sum Of The Sixth Powers Of Fibonacci Numbers H Ohtsuka, S Nakamura, *Congressus Numerantium* Vol. 201 (2010), Proceedings of the Thirteenth Conference on Fibonacci Numbers and their Applications , pp.297-300.

S Rabinowitz "Algorithmic Manipulation of Fibonacci Identities" in Applications of Fibonacci Numbers: Proceedings of the Sixth International Research Conference on Fibonacci Numbers and their Applications, editors G E Bergum, A N Philippou, A F Horodam; Kluwer Academic (1996), pages 389 - 408.

S Vajda, Fibonacci and Lucas numbers, and the Golden Section: Theory and Applications, Dover Press (2008).

This is a wonderful book, a classic, originally published in 1989 and now back in print in this Dover edition. This book is full of formulae on the Fibonacci numbers and Phi and the Lucas numbers. The whole book develops the formulae step by step, proving each from earlier ones or occasionally from scratch. It has a few errors in its formulae and all of them have been dutifully and exactly copied by authors such as Dunlap (see above) and others! Where I have identified errors, ***they have been corrected here on this page***.

Fibonacci Home Page

Fibonacci and Phi in the Arts

This is the first page on this Topic.

WHERE TO NOW???

This is the last Topic.
More topics at Ron Knott's Home page

Links and Bibliography