Fibonacci and Golden Ratio Formulae

Here are almost 300 formula involving the Fibonacci numbers and the golden ratio together with the Lucas numbers and the General Fibonacci series (the G series). This forms a major reference page for Ron Knott's Fibonacci Web site (http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fib.html) where there are many more details and explanations with applications, puzzles and investigations aimed at secondary school students and teachers as well as interested mathematical enthusiasts.

Note that it is easy to search for a named formula on this page since it is an HTML page and the formulae are not images. In your browser main menu, under the Edit menu look for Find... and type Vajda-N or Dunlap-N for the relevant formula. Full references are at the foot of this document.

A companion page on Linear Recurrences and their generating Functions for Fibonacci Numbers, Continued Fraction convergents, Pythagorean triples and other series of numbers.
1 Definitions and Notation

Beware of different golden ratio symbols used by different authors!
Where a formula below (or a simple re-arrangement of it) occurs in either Vajda or Dunlap's book, the reference number they use is given here. Dunlap's formulae are listed in his Appendix A3. Hoggatt's formulae are from his "Fibonacci and Lucas Numbers" booklet. Full bibliographic details are at the end of this page in the References section.

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\( \Phi = \frac{\sqrt{5} + 1}{2} = 1.6180339... \)  
Koshy uses \( \Phi \) (page 78)

\( \phi = \frac{\sqrt{5} - 1}{2} = 0.6180339... \)  
Koshy uses \( -\phi \) (page 78)

\( \text{abs}(x) = x \) if \( x \geq 0 \); \( \text{abs}(x) = -x \) if \( x < 0 \)
The absolute value of a number, its magnitude; ignore the sign;

\( \text{floor}(x) = \lfloor x \rfloor \)  
the nearest integer \( \leq x \).

When \( x > 0 \), this is "the integer part of \( x \)" or "truncate \( x \)"
i.e. delete any fractional part after the decimal point.  
3 = floor(3) = floor(3.1) = floor(3.9),  
-4 = floor(-4) = floor(-3.1) = floor(-3.9)

\( \text{round}(x) = \lceil x \rceil \)  
the nearest integer \( \geq x \);

i.e. delete any fractional part after the decimal point.  
3 = round(3) = round(3.1) = round(3.9),  
-4 = round(-4) = round(-3.9), -3 = round(-3.1)  
4 = round(3.5), -3 = round(-3.5)

\( \text{ceil}(x) = \lceil x \rceil \)  
the nearest integer \( \geq x \).

3 = ceil(3), 4 = ceil(3.1) = ceil(3.9),  
-3 = ceil(-3) = ceil(-3.1) = ceil(-3.9)

\( \text{frac}(x) = \text{frac}(x) \)  
the fractional part of \( x \), i.e. the part of \( \text{abs}(x) \)  
after the decimal point

3 = frac(3), 4 = frac(3.1) = frac(3.9),  
3 = frac(-3), 4 = frac(-3.1) = frac(-3.9)

\( \binom{n}{r} \)  
the element in row \( n \) column \( r \) of Pascal's Triangle  
the coefficient of \( x^r \) in \( (1+x)^n \)  
the number of ways of choosing \( r \) objects from a set of \( n \) different objects.  
\( n \geq 0 \) and \( r \geq 0 \) (otherwise value is 0)

<table>
<thead>
<tr>
<th>Formula</th>
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</table>
| \( F(0) = 0, F(1) = 1, \)  
\( F(n+2) = F(n+1) + F(n) \) | - | Definition of the Fibonacci series |
| \( F(-n) = (-1)^{n+1} F(n) \) | Vajda-2, Dunlap-5 | Extending the Fibonacci series 'backwards' |
| \( L(0) = 2, L(1) = 1, \)  
\( L(n+2) = L(n+1) + L(n) \) | - | Definition of the Lucas series |
| \( L(-n) = (-1)^n L(n) \) | Vajda-4, Dunlap-6 | Extending the Lucas series 'backwards' |
| \( G(n+2) = G(n+1) + G(n) \) | Vajda-3, Dunlap-4 | Definition of the Generalised Fibonacci series, G(0) and G(1) needed |
| \( \Phi = 1.618... = \frac{\sqrt{5} + 1}{2} \) | Dunlap-63 | Phi and \( -\phi \) are the roots of \( x^2 = x + 1 \) |

Fibonacci-type series with the rule \( S(i) = S(i-1) + S(i-2) \) for all integers \( i \):

<table>
<thead>
<tr>
<th>( i )</th>
<th>-6</th>
<th>-5</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>...</th>
</tr>
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<tbody>
<tr>
<td>Fibonacci ( F(i) )</td>
<td>...</td>
<td>-8</td>
<td>5</td>
<td>-3</td>
<td>2</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>Lucas ( L(i) )</td>
<td>...</td>
<td>18</td>
<td>-11</td>
<td>7</td>
<td>-4</td>
<td>3</td>
<td>-1</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>7</td>
<td>11</td>
<td>18</td>
</tr>
<tr>
<td>General Fib ( G(n,b) )</td>
<td>...</td>
<td>13a-8b</td>
<td>-8a+5b</td>
<td>5a-3b</td>
<td>-3a+2b</td>
<td>2a-b</td>
<td>-a+b</td>
<td>a</td>
<td>b</td>
<td>a+b</td>
<td>a+2b</td>
<td>2a+3b</td>
<td>3a+5b</td>
<td>5a+8b</td>
</tr>
</tbody>
</table>

file://Users/ronknott/MISC/WWW/Fibonacci/fibFormula...
phi = 0.618... = \frac{\sqrt{5} - 1}{2} \text{ \space Dunlap-65} \text{ Beware! Dunlap occasionally uses } \phi \text{ to represent our } phi = 0.61803.., \text{ but more frequently he uses } \phi \text{ to represent } -0.61803.. ![1]

## 2 Linear Formulae

Linear relationships involve only sums or differences of Fibonacci numbers or Lucas numbers or their multiples.

### 2.1 Linear Sums of Fibonacci numbers

- \[ F(n + 2) + F(n) + F(n - 2) = 4 F(n) \] \text{ B&Q(2003)-Identity 18}
- \[ F(n + 2) + F(n) = L(n + 1) \] \text{ by Definition of } L(n), \text{ Vajda-6, Hoggatt-I8, B&Q(2003) Identity 32, Dunlap-14, Koshy-5.14}
- \[ F(n + 2) - F(n) = F(n + 1) \] \text{ by Definition of } F(n)
- \[ F(n + 3) + F(n) = 2 F(n + 2) \] \text{ B&Q(2003)-Identity 16}
- \[ F(n + 3) - F(n) = 2 F(n + 1) \] \text{-}
- \[ F(n + 4) + F(n) = 3 F(n + 2) \] \text{ B&Q(2003)-Identity 17}
- \[ F(n + 2) + F(n - 2) = 3 F(n) \] \text{ B&Q(2003)-Identity 7}
- \[ F(n + 2) - F(n - 2) = L(n) \] \text{ Hoggatt-I14}
- \[ F(n + 4) - F(n) = L(n + 2) \] \text{-}
- \[ F(n + 5) + F(n) = F(n + 2) + L(n + 3) \] \text{-}
- \[ F(n + 5) - F(n) = L(n + 2) + F(n + 3) \] \text{-}
- \[ F(n + 6) + F(n) = 2 L(n + 3) \] \text{-}
- \[ F(n + 6) - F(n) = 4 F(n + 3) \] \text{-}
- \[ F(n) + 2 F(n - 1) = L(n) \] \text{(Dunlap-32), B&Q(2003) Identity 50}
- \[ F(n + 2) - F(n - 2) = L(n) \] \text{ Vajda-7a, Dunlap-15, Koshy-5.15}
- \[ F(n + 3) - 2 F(n) = L(n) \] \text{ possible correction for Dunlap-31}
- \[ F(n + 2) - F(n) + F(n - 1) = L(n) \] \text{ possible correction for Dunlap-31}
- \[ F(n) + F(n + 1) + F(n + 2) + F(n + 3) = L(n + 3) \] \text{ C Hyson(*)}

### 2.2 Linear Sums of Lucas numbers

- \[ L(n - 1) + L(n + 1) = 5 F(n) \] \text{ Vajda-5, Dunlap-13, Koshy-5.16, B&Q(2003)-Identity 34, Hoggatt-I9}
- \[ L(n) + L(n + 3) = 2 L(n + 2) \] \text{-}
- \[ L(n) + L(n + 4) = 3 L(n + 2) \] \text{-}
- \[ 2 L(n) + L(n + 1) = 5 F(n + 1) \] \text{ B&Q(2003)-Identity 52}
- \[ L(n + 2) - L(n - 2) = 5 F(n) \] \text{-}
- \[ L(n + 3) - 2 L(n) = 5 F(n) \] \text{-}

### 2.3 Linear Sum of a Fibonacci and a Lucas number

- \[ F(n) + L(n) = 2 F(n + 1) \] \text{ Vajda-7b, Dunlap-16, B&Q-Identity 51}
- \[ L(n) + 5 F(n) = 2 L(n + 1) \] \text{-}
2.4 Golden Ratio Formulae

Defining relations: The roots of $x^2 = x + 1$ are

$$\phi = \Phi = \frac{1 + \sqrt{5}}{2} = 1.61803398874994984820458683436...$$

$$-\phi = -\phi = \frac{1 - \sqrt{5}}{2} = -0.61803398874994984820458683436...$$

2.4.1 Basic Phi Formulae

$$\Phi \phi = 1$$ Vajda page 51(3), Dunlap-65

$$\Phi + \phi = \sqrt{5}$$ -

$$\Phi / \phi = \Phi + 1$$ -

$$\phi / \Phi = 1 - \phi$$ -

$$\Phi = \phi + 1 = \sqrt{5} - \phi$$ -

$$\phi = \Phi - 1 = \sqrt{5} - \Phi$$ -

$$\Phi^2 = \Phi + 1$$ Vajda page 51(4), Dunlap-64

$$\Phi^{n+2} = \Phi^{n+1} + \Phi^n \forall n \in \mathbb{Z}$$ Vajda page 51(4), Dunlap-64

$$\phi^2 = 1 - \phi$$ Vajda page 51(4), Dunlap-64

$$\phi^{n+2} = \phi^n - \phi^{n+1} \forall n \in \mathbb{Z}$$ Vajda page 51(4), Dunlap-64

$$\phi^n = \phi^{n+1} + \phi^{n+2} \forall n \in \mathbb{Z}$$ from line above

3 Golden Ratio with Fibonacci and Lucas

$$\Phi = \frac{\sqrt{5} + 1}{2} = \frac{1}{\phi} = \phi + 1; \quad \phi = \frac{\sqrt{5} - 1}{2} = \frac{1}{\Phi} = \Phi - 1$$

"Binet’s" Formula

De Moivre(1718), Binet(1843), Lamé(1844), Vajda-58, Dunlap-69, Hoggatt-page 11, B&Q(2003)-Identity 240

$$F(n) = \Phi^n - (-\phi)^n \sqrt{5}$$ Vajda-59, Dunlap-70, B&Q(2003)-Identity 241

$$L(n) = \Phi^n + (-\phi)^n$$ Vajda-59, Dunlap-70, B&Q(2003)-Identity 241

$$\Phi^n = \Phi F(n) + F(n-1)$$ Vajda-50a, Rabinowitz-28, B&Q(2003)-Corollary 33

$$\Phi^n = F(n+1) + F(n) \phi$$ Rabinowitz-28, B&Q(2003)-Corollary 33

$$\Phi^n = L(n) + F(n) \sqrt{5}$$ Vajda-50b, Rabinowitz-25, B&Q(2003)-Identity 242, I Ruggles (1963) FQ 1.2 pg 80

$$(-\phi)^n = \frac{L(n) - F(n) \sqrt{5}}{2}$$ Vajda-50c, I Ruggles (1963) FQ 1.2 pg 80, Rabinowitz-25, B&Q(2003)-Identity 243

$$(-\phi)^n = -\phi F(n) + F(n-1)$$ Rabinowitz-28

$$(-\phi)^n = F(n+1) - \Phi F(n)$$ Vajda-103b, Dunlap-75

$$\sqrt{5} \Phi^n = \Phi L(n) + L(n-1)$$ -

$$\sqrt{5} (-\phi)^n = \phi L(n) - L(n-1)$$ -

3.1 Some useful special cases

These follow simply from Vajda-50a and the basic definitions of Phi above.

$$\Phi + 2 = \sqrt{5} \Phi$$
\[ \Phi^2 + 1 = 2 + \Phi = \frac{5 + \sqrt{5}}{2} = \sqrt{5} \Phi \]

\[ \Phi^3 + 1 = 2 + 2 \Phi = 2 \Phi^2 = 3 + \sqrt{5} \]

\[ \Phi^4 + 1 = 5 + 8 \Phi = 9 + 4 \sqrt{5} \]

\[ \Phi^5 = 13 + 21 \Phi \]

3.2 Golden Ratio with Fibonacci and Lucas - Approximations

\[ \lim_{n \to \infty} \frac{F(n+1)}{F(n)} = \Phi \]

\[ \lim_{n \to \infty} \frac{F(n+m)}{F(n)} = \Phi^m \]

\[ F(n) = \text{round} \left( \frac{\Phi^n}{\sqrt{5}} \right), \text{if } n \geq 0 \]

\[ L(n) = \text{round}(\Phi^n), \text{if } n \geq 2 \]

\[ F(-n) = \text{round} \left( \frac{(-\Phi)^n}{\sqrt{5}} \right), \text{if } n \geq 0 \]

\[ L(-n) = \text{round}(\Phi^n), n \geq 2 \]

\[ F(n+1) = \text{round}(\Phi F(n), \text{if } n \geq 2) \]

\[ L(n+1) = \text{round}(\Phi L(n), \text{if } n \geq 4) \]

\[ \text{fract}(F(2n) \Phi) = 1 - \psi \]

\[ \text{fract}(F(2n+1) \Phi) = \psi \]

4 Fibonacci and Lucas Factors

F(nk) is a multiple of F(n)

\[ F(nk) \equiv 0 \pmod{F(k)} \]

\[ \gcd(F(m),F(n)) = F(\gcd(m,n)) \]

\[ F(nm+r) \equiv \pm F(r) \pmod{F(n)} \]

\[ \gcd(L(m),L(n)) = L(\gcd(m,n)) \]

\[ F(mq) = \sum_{j=1}^{q} F(m-1)^j F(m(q-j) + 1) \]

\[ \frac{F(kt)}{F(t)} = \sum_{i=0}^{(k-3)/2} (-1)^i L( (k-2i-1)t ) + (-1)^{(k-1)/2} \text{ for ODD } k \geq 3 \]

\[ \frac{F(kt)}{F(t)} = \sum_{i=0}^{(k-2)/2} (-1)^i L( (k-2i-1)t ) \text{ for EVEN } k \geq 2 \]

\[ \frac{L(kt)}{L(t)} = \sum_{i=0}^{(k-3)/2} (-1)^i L( (k-2i-1)t ) + (-1)^{(k-1)/2} \text{ for ODD } k \geq 3 \]

\[ L(t) \text{ is not a factor of } L(kt) \text{ for even } k \]
5 Order 2 Formulae

Order 2 means these formulae have terms involving the product of at most 2 Fibonacci or Lucas numbers.

5.1 Fibonacci numbers

\( F(n + 1)^2 + F(n)^2 = F(2n + 1) \)  
\( F(n + 1)^2 - F(n - 1)^2 = F(2n) \)
\( F(n^2 + 1)^2 + F(n^2)^2 = 3 F(n + 1)^2 - 2 (\pm 1)^n \)
\( F(n+3)^2 + F(n)^2 = 2 ( F(n+1)^2 + F(n+2)^2 ) \)
\( F(n + k + 1)^2 + F(n - k)^2 = F(2k + 1)F(2n + 1) \)
\( F(n^2 + 1)^2 + F(n^2)^2 = F(n + k - 1)^2 - F(n - k - 1)^2 - F(n + k + 1)F(2n + 2) \)
\( F(n + 1)F(n - 1) - F(n)^2 = (-1)^n \)
\( F(n)^2 - F(n + r)F(n - r) = (-1)^rF(r)^2 \)
\( F(n)F(m + 1) - F(m)F(n + 1) = (-1)^mF(n - m) \)
\( F(n + m) = F(n + 1)F(m + 1) - F(n - 1)F(m - 1) \)
\( F(n) = F(m)F(n + 1 - m) + F(m - 1)F(n) \)
\( F(n) = F(m)F(n + 1 - m) + F(m - 1)F(n - m) \)
\( F(n) = F(n - 1)F(n + 2) + (-1)^{n+1} \)
\( F(n + i)F(n + k) - F(n)F(n + i + k) = (-1)^k F(i)F(k) \)
\( 2 F(n + 1) = F(n) + \sqrt{5} F(n^2 + 4(-1)^n) \)
\( F(a)F(b) - F(c)F(d) = (-1)^r (F(a - r)F(b - r) - F(c - r)F(d - r)) \)
\( a + b = c + d \) for any integers \( a, b, c, d, r \)

Vajda-11, Dunlap-7, Lucas(1878), B&Q(2003)-Identity 13, Hoggatt-I11
Lucas(1878), B&Q(2003)-Identity 14, Hoggatt-I10
V E Hoggatt B-208 FQ 9 (1971) pg 217.
B&Q(2003)-Identity 30
Sharpe(1965), a generalization of Vajda-11, Dunlap-7
Melham(1999)
B&Q(2003)-Identity 8

Cassini's Formula (1680), Simson(1753), Vajda-29, Dunlap-9, Hoggatt-I13
special case of Catalan's Identity with \( r=1 \)
B&Q(2003)-Identity 8

Catalan's Identity (1879)
d'Ocagne's Identity,
special case of Vajda-9 with \( G=F \)
B&Q(2003)-Identity 231
alternative to Dunlap-10, B&Q(2003)-Identity 3;
variation of Hansen (1972)
Vorob'ev (1951) pages 9-10 proof, attributed to I S Sominskii
I Ruggles (1963) FQ 1.2 pg 77; Hoggatt-I25, Sharpe (1965)

5.2 Lucas numbers

\( F(kt) \sum_{i=0}^{k-2} (-1)^{i+1} F((k-2i-1)t) \) for EVEN \( k \geq 2 \)

Vajda-88

\( L(t) \) is not a factor of \( F(kt) \) for odd \( k \) and \( t \geq 3 \)

B&Q(2003) Identity 228

\( p \) prime \( \Rightarrow p \) is a factor of \( L(p) - 1 \)

B&Q(2003) Identity 229

\( p \) prime \( \Rightarrow p \) is a factor of \( L(2p) - 3 \)
\(L(n + 2) = 3L(n + 1) - L(n) + 10(-1)^n\)

from Vajda-17a

\(L(n + 1)^2 + L(n)^2 = L(2n) + L(2n + 2)\)

from Vajda-17a

\(L(n + 1)^2 - L(n - 1)^2 = L(2n + 1) + L(2n - 1)\)

from Vajda-17a

\(L(n + 1)L(n - 1) - L(n)^2 = -5(-1)^n\)

B&Q(2003)-Identity 60

\(L(2n) + 2(-1)^n = L(n)^2\)

Vajda-17c, Dunlap-12, B&Q(2003)-Identity 36

\(L(n + m) + (-1)^m L(n - m) = L(m)L(n)\)

Vajda-17a, Dunlap-11 (special cases: Hoggatt-I15, I18)

\(L(n + 1) - L(n - 1) = 5 F(2n)\)

Hoggatt-I15, special case of Vajda-17a

\(L(n + m) + (-1)^m L(n - m) = L(m) + 5 F(n)\)

Vajda-30, Vajda-31, Dunlap-27, Dunlap-30

\(F(n + 1)F(n) = F(n + m) + (-1)^m F(n - m)\)

Vajda-15a, Dunlap-19

\(L(n)F(m) = F(n + m) - (-1)^m F(n - m)\)

Vajda-15b, Dunlap-20

\(L(n)F(m) = F(n + m) + (-1)^m F(n - m)\)

Vajda-15b, Dunlap-20

\(5 F(m)F(n) = L(n + m) - 4(-1)^m L(n - m)\)

Vajda-16a, Dunlap-2, FQ (1967) B106 H H Ferns pp 466-467

5.3 Fibonacci and Lucas Numbers

\(F(n + k) + (-1)^k F(n-k) = F(n)L(k)\)

Bro U Alfred (1964), Bergum and Hoggatt (1975) eqns (5),(7)

\(F(n+k) - (-1)^k F(n-k) = L(n)F(k)\)

Bro U Alfred (1964), Bergum and Hoggatt (1975) equns (6),(8)

\(L(n+k) + (-1)^k L(n-k) = 5F(n)F(k)\)

Bro U Alfred (1964), Bergum and Hoggatt (1975) equns (10),(12)

\(2F(n + 1)L(n) = F(2n + 1) + (-1)^n\)

Vajda-13, Hoggatt-I7, Koshy-5.13, B&Q(2003)-Identity 33

\(F(n + 1) L(n) = F(2n + 1) + (-1)^n\)

Vajda-23, Dunlap-25

\(L(n + m) + (-1)^m L(n - m) = L(m) L(n)\)

Vajda-30, Vajda-31, Dunlap-27, Dunlap-30

\(F(n + 1) F(n) = F(n + m) + (-1)^m F(n - m)\)

Vajda-15a, Dunlap-19

\(L(n) F(m) = F(n + m) - (-1)^m F(n - m)\)

Vajda-15b, Dunlap-20

\(5 F(m) F(n) = L(n + m) - 4(-1)^m L(n - m)\)

Vajda-16a, Dunlap-2, FQ (1967) B106 H H Ferns pp 466-467

\(2F(n + 1) = L(n) + \sqrt{5} \sqrt{(L(n)^2 - 4(-1)^n)}\)

L(n+1) from L(n): Problem B-42, S Basin, FQ 2 (1964) page 329

Vajda-22, Dunlap-24
\[2 L(n + m) = L(m) L(n) + 5 F(n) F(m)\]

FQ (1967) B106 H H Ferns pp 466-467

\[-1^m \cdot 2 F(n - m) = L(m) F(n) - L(n) F(m)\]

Hansen (1972)

\[L(n + i) L(n + k) - L(n) L(n + i + k) = (-1)^{i+k} F(i) L(k)\]

Vajda-19a

\[L(n + k + 1) + L(n - k) = 5 F(2n + 1) F(2k + 1)\]

Melham (1999) Theorem 1

\[L(n + i) L(n + k) - L(n) L(n + i + k) = (-1)^{i+k} 5 F(i) F(k)\]

Vajda-20b

\[-1^m F(n - k) + (-1)^m F(k) F(n - m) + (-1)^n F(m) F(k - n) = 0\]

FQ 11 (1973) B228 page 108

\[-1^m L(n) F(n - k) + (-1)^n L(k) F(n - m) + (-1)^k L(m) F(k - n) = 0\]

FQ 11 (1973) B229 page 108

\[5 F(jk+r) F(ju+v) = L(j(k+u)+(r+v)) - (-1)^j F(j(k-u)+(r-v))\]

Hansen (1978)

\[F(jk+r) L(ju+v) = F(j(k+u)+(r+v)) + (-1)^j F(j(k-u)+(r-v))\]

Hansen (1978)

\[L(jk+r) L(ju+v) = L(j(k+u)+(r+v)) + (-1)^j L(j(k-u)+(r-v))\]

Hansen (1978)

\[5 F(a) F(b) - L(c) L(d) = (-1)^a (5 F(a - r) F(b - r) - L(c - r) L(d - r))\]

a+b=c+d for any integers a,b,c,d,r

Johnson

\[F(a) L(b) - F(c) L(d) = (-1)^b ( F(a-r) L(b-r) - F(c-r) L(d-r))\]

with a+b=c+d

Johnson-32, special case of Johnson-44

\[F(n+a+b) F(n-a) F(n-b) - F(n-a-b) F(n+a) F(n+b) = (-1)^{a+b} F(a) F(b) F(a+b) L(n)\]

Melham (2011) Theorem 1

\[F(n+a+b-c) F(n-a-c) F(n+b) - F(n-a-b+c) F(n+a) F(n+b)\]

\[(-1)^{a+b+c} F(a+b-c) F(c) F(n+a+b-c) + (-1)^c F(a-c) F(b-c) L(n)\]

Melham (2011) Theorem 5

\[F(i+j+k) = F(i+1) F(j+1) F(k+1) + F(i) F(j) F(k) - F(i-1) F(j-1) F(k-1)\]

for any integers i,j,k

Johnson's (6)

\[L(5n) = L(n) L(2n) + 5 F(n) + 3( L(2n) - 5 F(n) + 3), n \text{ odd}\]

Aurifeuille's Identity (1879)


6 Higher Order Fibonacci and Lucas

6.1 Fibonacci and Lucas cubed

\[F(3n) = F(n + 1)^3 + F(n)^3 - F(n-1)^3\]

Lucas (1876), B&Q(2003)-Identity 232

\[5 L(3n) = L(n + 1)^3 + L(n)^3 - 3 L(n-1)^3\]

Long (1986) equation (45)

\[3 F(3n) = F(n+2)^3 - 3 F(n)^3 + F(n-2)^3\]


\[L(3n) = L(n+1) F(n+1)^2 + L(n) F(n)^2 - L(n-1) F(n-1)^2\]

Long (1986) equation (43)

\[5 F(3n) = F(n+1) L(n+1)^2 + F(n) L(n)^2 - F(n-1) L(n-1)^2\]

Long (1986) equation (44)

\[F(n+1) F(n+2) F(n+6) - F(n+3)^3 = (-1)^n F(n)\]


\[F(n) F(n+4) F(n+5) - F(n+3)^3 = (-1)^n F(n+6)\]

The second is a variant with -n for n and using Vajda-2

\[F(n-2) F(n-1) F(n+3) - F(n)^3 = (-1)^n F(n-3)\]

Fairgrieve and Gould (2005) versions of the above two formulae of Melham

\[F(n+2) F(n+1) F(n-3) - F(n)^3 = (-1)^n F(n+3)\]

Fairgrieve and Gould (2005)

\[F(n+2) F(n+1)^2 - F(n)^3 = (-1)^n F(n-1)\]

Long (1986) equation (43)

\[F(n+2) F(n+1) F(n-3)^2 - F(n)^3 = (-1)^n F(n+1)\]

Long (1986) equation (44)

\[F(n+a+b) F(n-a) F(n-b) - F(n-a-b) F(n+a) F(n+b)\]

Melham (2011) Theorem 1

\[(-1)^{a+b} F(a) F(b) F(a+b) L(n)\]

Melham (2011) Theorem 5

6.2 Fibonacci and Lucas to the fourth

\[F(i+j+k) = F(i+1) F(j+1) F(k+1) + F(i) F(j) F(k) - F(i-1) F(j-1) F(k-1)\]

for any integers i,j,k

Johnson's (6)

\[L(5n) = L(n) L(2n) + 5 F(n) + 3( L(2n) - 5 F(n) + 3), n \text{ odd}\]

Aurifeuille's Identity (1879)

\[
F(n-1)F(n+1)^2 - F(n-2)F(n+2)^2 = 4(-1)^n F(n)^2
\]
\[
F(n-3)F(n-1)F(n+1)F(n+3) - F(n)^4 = (-1)^n L(n)^2
\]
\[
F(n)^2 F(m+1) F(m-1) - F(m)^2 F(n+1) F(n-1) = (-1)^{m+n} F(m+n) F(m-n)
\]
\[
F(n-2)F(n-1)F(n+1)F(n+2) + 1 = F(n)^4
\]
\[
F(n+a+b+c)F(n-a)F(n-b)F(n-c) - F(n-a-b-c)F(n+a)F(n+b)F(n+c) = (-1)^{a+b+c} F(a+b)F(a+c)F(b+c)F(2n)
\]
\[
F(n+a+b+c-d)F(n-a+d)F(n-b+d)F(n-c+d) - F(n-a-b-c+2d)F(n+a)F(n+b)F(n+c) = (-1)^{a+b+c+d} F(a+b+d)F(a+c+d)F(b+c+d)F(2n+d)
\]
\[
\left( F(n+1)^2 - (F(n)F(n+1))^2 \right)^2 = (F(n+1)F(n+2) - F(n-1)F(n))^2
\]
\[
2 \left( F(n)^3 + F(n+1)^3 + F(n+2)^3 \right) = \left( 5F(2n+1) \right)^2
\]
\[
L(n-1)L(n+2) + 2L(n)L(n+1) = 5F(2n+1)
\]

### 6.3 Fibonacci and Lucas Higher Powers

\[
F(n)F(n+1)F(n+2)F(n+4)F(n+5)F(n+6) + L(n+3)^2 = (F(n+3) F(n+2) F(n+4) - F(n+3)^2)^2
\]

Melham (2011) 21

Morgado (1987)

\[
\left( \frac{L(n) + \sqrt{5} F(n)}{2} \right) = \frac{L(kn) + \sqrt{5} F(kn)}{2}
\]

De Moivre Analogue,

S Fisk (1963) FQ 1.2 Problem B-10, pg 85.

Hoggatt-I44

### 7 Fibonomial formulae

The Fibonomials are defined using Fibonacci numbers instead of integers in binomial coefficients and Fibonacci factorials instead of normal factorials. There are many analogues results to those using binomial coefficients but using Fibonomials instead.

We define \( F(n) = F(n)F(n-1)...F(2)F(1) \) for which some authors use \( n \) for \( n! \), to compare with \( n! = n(n-1)...3.2.1 \).

There is no universal notation for the Fibonomial. The fibonomial "Fibonacci n choose k" is defined as:

\[
\binom{n}{k} = \frac{F(n)}{F(k)F(n-k)} \quad \text{if } n \geq k \geq 0.
\]

Vajda (page 74) uses \( J(n,k) \). D Knuth and others use double brackets: \( n \) while Melham (1999) and Vajda-32

Others use square brackets: \( n \)

A simple alternative is to write \( \text{fibonomial}(n,k) \).

Here is a table of some values of the fibonomial (A010048)

<table>
<thead>
<tr>
<th>n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

file:///Users/ronknott/MISC/WWW/Fibonacci/fibForm...
\[
\begin{array}{c}
\binom{m+n}{n}_F = F(m-1)\binom{m+n-1}{n-1}_F + F(n+1)\binom{m+n-1}{n}_F \\
\sum_{j=0}^{m} (-1)^{j+3/2} \binom{m}{j}_F F(n+m-j)^{m+1} = F(m) F((m+1)n+m(m+1)/2))
\end{array}
\]

Vajda page 74, "add the two numbers above" analogy from Pascal's triangle

Melham (1999)...

\[
\begin{array}{c}
\sum_{k=0}^{p} \binom{p}{k} (-1)^{k/2} F(n-k)^{p-1} = 0, \text{ if } p>0 \\
F(k) \binom{n}{k}_F = F(n-k+1) \binom{n}{k-1}_F \\
F(n-k) \binom{n}{k-1}_F = F(n) \binom{n-1}{k-1}_F \\
\binom{n}{k}_F \binom{k}{j}_F = \binom{n-j}{k-j}_F
\end{array}
\]

Knuth AoCP Vol 1 section 1.2.8 Exercise 30, (1997)

\[
\begin{array}{c}
0 = F(n) - F(n-1) - F(n-2) \\
0 = F(n)^2 - 2 F(n-1)^2 - 2 F(n-2)^2 + F(n-3)^2 \\
0 = F(n)^3 - 3 F(n-1)^3 - 6 F(n-2)^3 - 3 F(n-3)^3 + F(n-4)^3 \\
0 = F(n)^4 - 5 F(n-1)^4 - 15 F(n-2)^4 - 15 F(n-3)^4 + 5 F(n-4)^4 - F(n-5)^4
\end{array}
\]

Brousseau (1968)...but the general formula was not given. For this see next line:

\[
\begin{array}{c}
0 = F(n)^5 - 2 F(n-1)^5 - 15 F(n-2)^5 + 5 F(n-3)^5 + 15 F(n-4)^5 - 2 F(n-5)^5 \\
0 = F(n)^6 - 3 F(n-1)^6 - 15 F(n-2)^6 + 3 F(n-3)^6 + 15 F(n-4)^6 - 6 F(n-5)^6 + F(n-6)^6
\end{array}
\]

8 G Formulae

G(i) is the General Fibonacci series. It has the same recurrence relation as Fibonacci and Lucas, namely \(G(n+2) = G(n+1) + G(n)\) for all integers \(n\) (i.e. \(n\) can be negative) Vajda-3,Dunlap 4, but the "starting values" of \(G(0)=a\) and \(G(1)=b\) can be specified. To make it clear which starting values for \(G(0)=a\) and \(G(1)=b\) are being used, we write \(G(a,b,i)\) for \(G(i)\). \(G(n)\) is an abbreviation for \(G(a,b,n)\) when \(a\) and \(b\) are understood from the context. Special cases are the Fibonacci and Lucas series since \(F(n) = G(0,1,n)\) and \(L(n)=G(2,1,n)\):
8.1 Basic G Formulae

Two independent G series are here denoted G(n) and H(n), i.e. G(0) and G(1) are independent of H(0) and H(1).

\[ G(n) = G(0) F(n-1) + G(1) F(n) \]

B&Q(2003)-Identity 37

\[ G(-n) = (-1)^{m} (G(0) F(n + 1) - G(1) F(n)) \]

ditto - applying Vajda-2 or Vajda-9 with \( n = 0 \)

\[ \sqrt{5} G(n) = (G(0) \phi + G(1)) \phi^{n} + (G(0) \phi - G(1)) (-\phi)^{n} \]

Vajda-55/56, Dunlap-77, B&Q(2003)-Identity 244

\[ F(n) = \frac{G(0) G(n+1) - G(1) G(n)}{G(0) G(2) - G(1)^{2}} \]

Vajda-55/56, Dunlap-77, B&Q(2003)-Identity 244

\[ 2 G(k) = (2 G(1) - G(0)) F(k) + G(0) L(k) \]

Johnson-46

\[ G(n + m) = F(m - 1) G(n) + F(m) G(n + 1) \]

Vajda-8, Dunlap-33, B&Q(2003)-Identity 38, Johnson-40

\[ G(n - m) = (-1)^{m} (F(m + 1) G(n) - F(m) G(n + 1)) \]

Vajda-9, Dunlap-34, B&Q(2003)-Identity 47

\[ G(n + m) + (-1)^{m} G(n - m) = L(m) G(n) \]

Vajda-10a, Dunlap-35, B&Q(2003)-Identity 45, Bergum & Hoggatt (1975) (36) and (38)

\[ G(n + m) - (-1)^{m} G(n - m) = F(m) (G(n-1) + G(n+1)) \]

Vajda-10b, Dunlap-36, B&Q(2003)-Identity 48, Bergum & Hoggatt (1975) (37) and (39)

\[ G(m) F(n) - G(n) F(m) = (-1)^{m+1} G(m) F(n - m) \]

Vajda-21a

\[ G(m) F(n) - G(n) F(m) = (-1)^{m} G(0) F(n - m) \]

Vajda-21b

\[ G(m+k) F(n+k) + (-1)^{k+1} G(m) F(n) = F(k) G(m + n + k) \]

Howard(2003)

8.2 G Formulae of Order 2 or more

These formulae include terms which are a product of two G numbers either from the same G series of from two different G series i.e. with different index 0 and 1 values. Where the series may be different they are denoted G and H e.g. special cases include G = F (i.e. Fibonacci) and H = L (i.e. Lucas), or they could also be the same series G = H.

\[ G(n + i) H(n + k) - G(n) H(n + i + k) \]

Vajda-18 (corrected), B&Q(2003)-Identity 44 (also Identity 68)

a special case of Johnson-44:

\[ p, q, r, s, n \text{ are integers} \]

\[ G(n + 1) G(n - 1) - G(n)^{2} = (-1)^{p} (G(1)^{2} - G(0) G(2)) \]

Vajda-28, B&Q(2003)-Identity 46

\[ 4 G(n-1)G(n) + G(n-2)^{2} = G(n+1)^{2} \]

B&Q(2003)-Identity 65

\[ G(n + 3)^{2} + G(n)^{2} = 2 (G(n+1)^{2} + G(n+2)^{2}) \]

B&Q(2003)-Identity 70

\[ G(i+j+k) = F(i+1)F(j+1)G(k+1) + F(i)F(j)G(k) - F(i-1)F(j-1)G(k-1) \]

for any integers \( i, j, k \)

Johnson (39a)

\[ 4G(i)^{2}G(i+1)^{2} + G(i-1)^{2}G(i+2)^{2} = (G(i)^{2} + G(i+1)^{2})^{2} \]

Generalised Fibonacci Pythagorean Triples

Horadam (1967)

\[ G(n + 2)G(n + 1)G(n - 1)G(n - 2) + (G(2)G(0) - G(1)^{2})^{2} \]

B&Q(2003)-Identity 59

9 Summations

This section has formulae that sum a variable number of terms.
9.1 Fibonacci and Lucas Summations

These formulae involve a sum of Fibonacci or Lucas numbers only.

\[ \sum_{i=0}^{n} F(i) = F(n + 2) - 1 \]
Hoggatt-I1, Lucas(1878), B&Q 2003-Identity 1

\[ \sum_{i=0}^{n} (-1)^i F(i) = (-1)^{n+1} F(n - 1) - 1 \]
B&Q 2003-Identity 21

\[ \sum_{i=0}^{n} L(i) = L(n + 2) - 1 \]
Hoggatt-I2

\[ \sum_{i=a}^{n} F(i) = F(n + 2) - F(a + 1) \]
- ...

\[ \sum_{i=a}^{n} L(i) = L(n + 2) - L(a + 1) \]
- ...

\[ \sum_{i=0}^{n} F(2i) = F(2n + 1) - 1, n \geq 0 \]
Hoggatt-I6, Lucas(1878), B&Q(2003)-Identity 12

\[ \sum_{i=1}^{n} F(2i - 1) = F(2n), n \geq 1 \]
Hoggatt-I5, Lucas(1878), B&Q(2003)-Identity 2

\[ \sum_{i=1}^{n} L(2i - 1) = L(2n) - 2 \]
-

\[ \sum_{i=1}^{n} 2^{n-i} F(i-1) = 2^n - F(n+2) \]
Vajda-37a(adapted), Dunlap-42(adapted), B&Q(2003)-Identity 10

\[ \sum_{i=0}^{n} 2^i L(i) = 2^{n+1} F(n + 1) \]
B&Q(2003)-Identity 236

\[ \sum_{i=0}^{n} F(3i + 1) = \frac{F(3n + 3)}{2} \]
B&Q(2003)-Identity 23

\[ \sum_{i=0}^{n} F(3i + 2) = \frac{F(3n + 4) - 1}{2} \]
B&Q(2003)-Identity 24 (corrected)

\[ \sum_{i=0}^{n} F(3i) = \frac{F(3n + 2) - 1}{2} \]
B&Q(2003)-Identity 25 (corrected)

\[ \sum_{i=0}^{n} F(4i) = F(2n + 1)^2 - 1 \]
B&Q 2003-Identity 27

\[ \sum_{i=0}^{n} F(4i + 1) = F(2n + 1)F(2n + 2) \]
B&Q 2003-Identity 26

\[ \sum_{i=0}^{n} F(4i + 2) = F(2n + 1)F(2n + 3) - 1 \]
B&Q 2003-Identity 29

\[ \sum_{i=0}^{n} F(4i + 3) = F(2n + 3)F(2n + 2) \]
B&Q 2003-Identity 28

\[ \sum_{i=0}^{n} (-1)^i L(n - 2i) = 2 F(n + 1) \]
Vajda-97, Dunlap-54
\[ \sum_{i=0}^{n} (-1)^i L(2n - 2i + 1) = F(2n + 2) \quad \text{B&Q(2003)-Identity 55} \]

### 9.2 Decimal (and other bases) fractions

We saw in *The Fibonacci Series as a Decimal Fraction* that the Fibonacci series occurs naturally as the decimal expansion of a simple fraction in several ways:

\[
\begin{align*}
\frac{1}{89} & = 0.011235\ldots \\
\frac{1}{9899} & = 0.000101020305081321\ldots
\end{align*}
\]

with a varying number of decimal digits before the Fibonacci numbers overlap and the series is obscured. This section gives formulae for these fractions for various subsequences of Fibonacci and General Fibonacci series.

\[
\sum_{i=1}^{\infty} 10^{-mt+k} F(ak) = \frac{F(a)}{10^m - 10^m L(a) - (-1)^a} \quad \text{Hudson and Winans (1981)}
\]

If \( P(n) = a P(n-1) + b P(n-2) \) for \( n \geq 2; \ P(0) = c; \ P(1) = d \) and \( m \) and \( N \) are defined by \( B = m + Ba + b, \ N = cm + dB + bc, \)
then

\[
\frac{N}{Bm} = \sum_{i=1}^{\infty} \frac{P(i-1)}{B} \quad \text{Long (1981)}
\]

provided that \( |(a+\sqrt{(a^2+4b)}/(2B))| < 1 \) and
\( |(a-\sqrt{(a^2+4b)}/(2B))| < 1 \)

### 9.3 Summations with fractions

\[
\begin{align*}
\sum_{i=0}^{\infty} \frac{F(i)}{2} & = 2 \quad \text{Vajda-60, Dunlap-51} \\
\sum_{i=0}^{\infty} \frac{L(i)}{2} & = 6 \\
\sum_{i=0}^{\infty} \frac{F(i)}{r^i} & = \frac{r}{r^2 - r - 1} \\
\sum_{i=0}^{\infty} \frac{L(i)}{r^i} & = 2 + \frac{r+2}{r^2 - r - 1} \\
\sum_{i=1}^{\infty} \frac{i F(i)}{2} & = 10 \quad \text{Vajda-61, Dunlap-52} \\
\sum_{i=1}^{\infty} \frac{i L(i)}{2} & = 22 \\
\sum_{i=0}^{\infty} \frac{1}{F(2^i)} & = 4 - \Phi = 3 - \phi \quad \text{Vajda-77(corrected), Dunlap-53(corrected)} \\
\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{F(2^i r)} & = \frac{(-1)^i F(r(2^{n-1}))}{F(r) F(r^n)} \quad \text{Vajda-89 (corrected)}
\end{align*}
\]
\[
\sum_{k \geq 2} \frac{1}{F(k-1)F(k+1)} = 1 \quad \text{R L Graham (1963) FQ 1.1, Problem B-9, pg 85, FQ 1.4 page 79}
\]

\[
\sum_{k \geq 2} \frac{F(k)}{F(k-1)F(k+1)} = 2 \quad \text{R L Graham (1963) FQ 1.1, Problem B-9, pg 85}
\]

\[
\sum_{k \geq 2} \frac{(-1)^k}{F(k)F(k-1)} = \phi \quad \text{Johnson-11, Vajda-102}
\]

\section*{9.4 Order 2 summations}

\[
\sum_{i=1}^{n} F(i)^2 = F(n) F(n + 1) \quad \text{Vajda-45, Dunlap-5, Hoggatt-I3, Lucas(1878), Koshy-77, B&Q(2003)-Identity 9 (Identity 233 variant)}
\]

\[
\sum_{i=1}^{n} L(i)^2 = L(n) L(n + 1) - 2 \quad \text{Hoggatt-I4}
\]

\[
\sum_{i=0}^{2n-1} i^2 = 5 F(2n) F(2n - 1) \quad -
\]

\[
\sum_{i=1}^{2n} F(i) F(i - 1) = F(2n)^2 \quad \text{Vajda-40, Dunlap-45}
\]

\[
\sum_{i=1}^{2n} L(i) L(i - 1) = L(2n)^2 - 4 \quad -
\]

\[
\sum_{i=1}^{2n+1} F(i) F(i - 1) = F(2n + 1)^2 - 1 \quad \text{Vajda-42, Dunlap-47}
\]

\[
\sum_{i=1}^{2n+1} L(i) L(i - 1) = L(2n + 1)^2 - 1 \quad -
\]

\[
5 \sum_{k=0}^{n} (-1)^{r(1+k)} F(r(1+k))^2 = (-1)^r(n+1) \frac{F((2n+3)r)}{F(r)} - 2n - 3 \quad \text{Vajda-93}
\]

\[
\sum_{k=0}^{n} (-1)^{r(1+k)} L(r(1+k))^2 = (-1)^r(n+1) \frac{F((2n+3)r)}{F(r)} + 2n + 1 \quad \text{Vajda-94}
\]

\[
\sum_{i=0}^{n-1} F(2i + 1)^2 = \frac{F(4n) + 2n}{5} \quad \text{Vajda-95, B&Q(2003)-Identity 234}
\]

\[
\sum_{i=0}^{n} F(2i)^2 = \frac{F(4n + 2) - 2n - 1}{5} \quad \text{Vajda page 70}
\]

\[
\sum_{i=0}^{n-1} L(2i + 1)^2 = F(4n) - 2n \quad \text{Vajda-96, B&Q(2003)-Identity 54}
\]

\[
\sum_{i=1}^{n} L(2i)^2 = F(4n + 2) + 2n - 1 \quad \text{Vajda page 70}
\]
\[ 5 \sum_{i=0}^{n} F(i) F(n-i) = (n+1) L(n) - 2 F(n+1) \]  
\[ n \sum_{i=0}^{n} L(i) L(n-i) = (n+1) L(n) - 2 F(n+1) \]

9.5 Summations of order > 2

\[ 10 \sum_{i=1}^{n} F(i)^3 = F(3n+2) + 6(-1)^{n+1} F(n-1) + 5 \]  
adapted from Benjamin, Carnes, Cloitre (2009)

\[ 25 \sum_{i=1}^{n} F(i)^4 = F(4n+2) + 4(-1)^{n+1} F(2n+1) + 6n + 3 \]  
see A005969

\[ 4 \sum_{k=1}^{n} F(k)^6 = F(n)^5 F(n+3) + F(2n) \]  
Ohtsuka and Nakamura (2010) Theorem 1

\[ 4 \sum_{k=1}^{n} L(k)^6 = L(n)^5 L(n+3) + 125 F(2n) - 128 \]  
Ohtsuka and Nakamura (2010) Theorem 2

9.6 G Summations

Two independent G series are denoted G(n) and H(n).

\[ \sum_{i=1}^{n} G(i) = G(n+2) - G(2) \]  
L G Brökling (1964) FQ 2.1 Problem B-20 solution, pg76; Vajda-33; Dunlap-38; B&Q(2003)-Identity 39

\[ \sum_{i=a}^{n} G(i) = G(n+2) - G(a+1) \]  
-

\[ \sum_{i=1}^{n} G(2i-1) = G(2n) - G(0) \]  
Vajda-34, Dunlap-37, B&Q(2003)-Identity 61

\[ \sum_{i=1}^{n} G(2i) = G(2n+1) - G(1) \]  

\[ \sum_{i=1}^{n} G(2i) - \sum_{i=1}^{n} G(2i-1) = \sum_{i=1}^{2n} (-1)^i G(i) = G(2n-1) + G(0) - G(1) \]  
Vajda-36, Dunlap-40

\[ \sum_{k=1}^{n} G(k-1) 2^{-k} = (G(0) + G(3))/2 - G(n+2) 2^{-n} \]  
Vajda-37, Dunlap-41, B&Q(2003)-Identity 69

\[ \sum_{i=1}^{4n+2} G(i) = L(2n+1) G(2n+3) \]  
Vajda-38, Dunlap-43, B&Q(2003)-Identity 49

\[ \sum_{i=1}^{2n} G(i) G(i-1) = G(2n)^2 - G(0)^2 \]  
Vajda-39, Dunlap-44, B&Q(2003)-Identity 41
\[
\sum_{i=1}^{2n+1} G(i) G(i-1) = G(2n+1)^2 - G(0)^2 - G(1)^2 + G(0) G(2)
\]
Vajda-41, Dunlap-46

\[
\sum_{i=1}^{n} G(i+2) G(i-1) = G(n+1)^2 - G(1)^2
\]
Vajda-43, Dunlap-48, B&Q(2003)-Identity 64

\[
(1 + (-1)^i L(r)) \sum_{k=0}^{n} G(m + kr) = \\
G(m) - G(m+(n+1)r) + (-1)^i (G(m+nr) - G(m-r))
\]

\[
\sum_{i=1}^{n} G(i)^2 = G(n) G(n + 1) - G(0) G(1)
\]

\[
\sum_{i=0}^{\infty} \frac{G(a, b, i)}{r^i} = a + \frac{a + b r}{r - r - 1}
\]

\[
\sum_{i=0}^{\infty} \frac{r (b r^2 - 2 a r + b - a)}{(r - r - 1)^2} = \frac{a + b r}{r - r - 1}
\]

\[
\sum_{i=1}^{2n-1} G(i) H(i) = G(2n) H(2n-1) - G(0) H(1)
\]

Fibonacci with a Golden Ring

\[
\sum_{i=1}^{n} \binom{n-i}{i-1} = F(n)
\]
B&Q(2003) Identity-4

\[
\sum_{i=0}^{\infty} \binom{n-i-1}{i} = F(n)
\]
Vajda-54(corrected), Dunlap-84(corrected)

\[
\sum_{i=0}^{n} \binom{n+i}{2i} = F(2n + 1)
\]
B&Q(2003)-Identity 165

\[
\sum_{i=0}^{n-1} \binom{n+i}{2i+1} = F(2n)
\]
B&Q(2003)-Identity 166

\[
\sum_{k=0}^{n} \binom{n}{k} F(k) = F(2n)
\]
S Basin & V Ivanoff (1963) Problem B-4, FQ 1.1 pg 74, FQ1.2 pg 79; B&Q(2003)-Identity 6

\[
\sum_{k=0}^{n} \binom{n}{k} (-1)^{k+1} F(k) = F(n)
\]
I Ruggles (1963) FQ 1.2 pg 77

\[
\sum_{k=0}^{n} \binom{n}{k} (-1)^{\frac{k}{2}} L(k) = L(n)
\]
I Ruggles (1963) FQ 1.2 pg 77

\[
\sum_{k=0}^{n} \binom{n}{k} F(p-k) = F(p+n)
\]
B&Q(2003)-Identity 20

9.7 Summations with Binomial Coefficients
\[ \sum_{k=0}^{n} \binom{n}{k} 2^k F(k) = F(3n) \]


\[ \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} 2^k F(2k) = F(3n) \]

Griffiths (2013) page 239-corrected

\[ \sum_{k=0}^{n} \binom{n}{k} F(3k + m) = 2^n F(2n + m) \]

Griffiths (2013)

\[ \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} F(3k) = F(n) \]

Griffiths (2013) page 239

\[ \sum_{i=0}^{n} \binom{n+1}{i+1} F(i) = F(2n + 1) - 1 \]

Vajda-50, Dunlap-82

\[ \sum_{i=0}^{2n} \binom{2n}{i} F(2i + p) = 5^n F(2n + p) \]

Hoggatt-I41 (special case p=0: Vajda-69, Dunlap-85)

\[ \sum_{i=0}^{2n} \binom{2n}{i} L(2i) = 5^n L(2n) \]

Vajda-71, Dunlap-87

\[ \sum_{i=0}^{2n+1} \binom{2n+1}{i} F(2i + p) = 5^n L(2n + 1 + p) \]

Hoggatt-I42 (special case p=0: Vajda-70, Dunlap-86)

\[ \sum_{i=0}^{2n+1} \binom{2n+1}{i} L(2i) = 5^{n+1} F(2n + 1) \]

Vajda-72, Dunlap-88

\[ \sum_{i=0}^{2n} \binom{2n}{i} F(i)^2 = 5^{n-1} L(2n) \]

Vajda-73, Dunlap-89,Hoggatt-I45

\[ \sum_{i=0}^{2n} \binom{2n}{i} L(i)^2 = 5^n L(2n) \]

Vajda-75, Dunlap-91, Hoggatt-I46

\[ \sum_{i=0}^{2n+1} \binom{2n+1}{i} F(i)^2 = 5^n F(2n + 1) \]

Vajda-74, Dunlap-90, Hoggatt-I47

\[ \sum_{i=0}^{2n+1} \binom{2n+1}{i} L(i)^2 = 5^{n+1} F(2n + 1) \]

Vajda-76, Dunlap-92

\[ \sum_{i=0}^{\infty} 5^i \binom{n}{2i+1} = 2^{n-1} F(n) \]

Vajda-91, B&Q(2003)-Identity 235, Catalan 1857

\[ \sum_{i=0}^{\infty} 5^i \binom{n}{2i} = 2^{n-1} L(n) \]

Vajda-92, B&Q(2003)-Identity 237, Catalan (1857)-see Vajda pg 69

\[ \sum_{i=0}^{k} \binom{k}{i} F(n) F(n-1) k-i F(i) = F(kn) \]

Rabinowitz-17 (special case of Vajda-66)
\[ \sum_{i=0}^{k} (\binom{k}{i}) F(n) F(n-1)^{k-i} L(i) = L(kn) \quad \text{Rabinowitz-17 (special case of Vajda-66)} \]

\[ \sum_{i=0}^{p} (\binom{p}{i}) F(t) F(t-1)^{p-i} G(m+i) = G(m+tp) \quad \text{Vajda-66, B&Q(2003) Identity-11} \]

\[ \sum_{i \geq 0} \sum_{j \geq 0} \binom{n-i}{j} \binom{n-j}{i} = F(2n+3) \quad \text{B&Q(2003) Identity 5} \]

\[ \sum_{i=0}^{n} (-1)^i \binom{2n+1}{i} L(2n+1-2i) = 1 \quad \text{Griffiths (2013)} \]

9.8 Powers of Fibonacci and Lucas as Sums

\[ 5^{\frac{k}{2}} F(t)^k = \sum_{i=0}^{(k-1)/2} \binom{k}{i} (-1)^{i+1} 5^i F((k-2i)t), k \text{ odd} \quad \text{Vajda-80} \]

\[ 5^{\frac{k}{2}} F(t)^k = \sum_{i=0}^{k/2-1} \binom{k}{i} (-1)^{i+1} L((k-2i)t) + \binom{k}{k/2} (-1)^{i+1/2} k \quad \text{Vajda-81} \]

\[ L(t)^k = \sum_{i=0}^{(k-1)/2} \binom{k}{i} (-1)^i L((k-2i)t), k \text{ odd} \quad \text{Vajda-78} \]

\[ L(t)^k = \sum_{i=0}^{k/2-1} \binom{k}{i} (-1)^i L((k-2i)t) + \binom{k}{k/2} (-1)^{i+1} k \quad \text{Vajda-79} \]

\[ F_m F_n = (-1)^{kr} \sum_{h=0}^{k} \binom{k}{h} (-1)^h F_r F_t F_r^{k-h} F_{r+m}^{n+kr+hm} \quad \text{On a General Fibonacci Identity} \]

9.9 Summations with Binomials and G Series

\[ \sum_{i=0}^{n} \binom{n}{i} G(i) = G(2n) \quad \text{I Ruggles (1963) FQ 1.2 pg 77; Vajda-47; Dunlap-80} \]

\[ \sum_{i=0}^{n} \binom{n}{i} 2^i G(i) = G(3n) \quad \text{B&Q(2003)-Identity 239} \]

\[ \sum_{i=0}^{n} \binom{n}{i} G(p-i) = G(p+n) \quad \text{Vajda-46, Dunlap-79, B&Q(2003)-Identity 40} \]

\[ \sum_{i=0}^{n} \binom{n}{i} G(p+i) = G(p+2n) \quad \text{Vajda-49, Dunlap-81} \]
\[ \sum_{i=0}^{p} (-1)^{p-i} \binom{p}{i} G(n+i) = G(n-p) \quad \text{Vajda-51, Dunlap-83} \]

## 10 Products

\[ \frac{F(n+1)}{F(n+2)} = \prod_{k=1}^{n} 1 - \frac{(-1)^k}{F(k+1)^2}, \quad n \geq 0 \quad \text{B&Q(2003) Identity 22} \]

## 11 Trigonometric Formulae

\[ F(n) = \prod_{k=1}^{1(n-1)/2} 3 + 2 \cos \left( \frac{2k\pi}{n} \right), \quad n \geq 1 \quad \text{D Lind, Problem H-93, FQ 4 (1966), page 332} \]

\[ L(n) = \prod_{k=0}^{1(n-2)/2} 3 + 2 \cos \left( \frac{(2k+1)\pi}{n} \right), \quad n \geq 2 \quad \text{D Lind, Problem H-93, FQ 4 (1966), page 252, corrected page 332} \]

## 12 E and Logs

Here we use \( g \) for \( \ln(\Phi) \), the natural log of \( \Phi \) so that \( \cosh(g) = \sqrt{5} / 2 \).

\[ F(2n) = \frac{2}{\sqrt{5}} \sinh(2ng) \quad \text{from Binet's formula} \]

\[ F(2n+1) = \frac{2}{\sqrt{5}} \cosh((2n+1)g) \quad \text{from Binet's formula} \]

\[ L(2n) = 2 \cosh(ng) \quad \text{from Binet's formula} \]

\[ L(2n+1) = 2 \sinh(ng) \quad \text{from Binet's formula} \]

\[ \sum_{k=1}^{\infty} \frac{\Phi F(k) - F(k+1)}{k} = \sum_{k=1}^{\infty} \frac{\sqrt{5} F(k) - L(k)}{2k} = g \quad \text{C. Brown (Jan 2016) private communication} \]

\[ \sum_{n=0}^{\infty} \frac{F(n)}{n!} = \frac{e^\Phi - e^{-\Phi}}{\sqrt{5}} \quad \text{Exponential Generating Functions For Fibonacci Identities} \]

\[ \sum_{n=0}^{\infty} \frac{L(n)}{n!} = e^\Phi + e^{-\Phi} \quad \text{Exponential Generating Functions For Fibonacci Identities} \]

## 13 Complex Numbers

\[ i = \sqrt{-1} \]

\[ \sin(\pi/2 + i \ln(\Phi)) = (\sqrt{5}/2) - \Phi + \frac{1}{2} \quad \text{Schroeder 1986, equation (5.41) page 68} \]
\[ F(n) = \prod_{k=1}^{n-1} \left( 1 - 2 \cos \frac{n}{n} \right) \quad \text{D Lind, Problem H-64, FQ 3 (1965), page 116} \]

\[ F(n) = \frac{2 i^{-n}}{\sqrt{5}} \sin (-i n \ln (i \Phi)) \quad \text{from Rabinowitz-7 corrected, using } \Phi^2 = (\sqrt{5} + 1)/(\sqrt{5} - 1) \]

\[ F(n) = \frac{2 i^{-n}}{\sqrt{5}} \sinh (n \ln (i \Phi)) \quad \text{from Rabinowitz-7 corrected} \]

\[ L(n) = 2 i^{-n} \cos (-i n \ln (i \Phi)) \quad \text{from Rabinowitz-7 corrected} \]

\[ L(n) = 2 i^{-n} \cosh (n \ln (i \Phi)) \quad \text{from Rabinowitz-7 corrected} \]

\[ \sqrt{1 + 2 i} = \Phi + i \phi \]

\[ = [1 + i; 2 + 2i] \]

\[ I J \text{ Good (1993)} \]

\[ \sqrt{1 + i/2} = (\sqrt{\sqrt{5} + 2 + i \sqrt{\sqrt{5} - 2}})/2 \]

\[ = (\Phi^{3/2} + i \phi^{3/2})/2 \]

\[ = [1 + i/2 ; 1 + i] \]

\[ \sqrt{1 + i} = \sqrt{\Phi + i \phi} \]

### 14 Generating Functions

This section is now part of the following reference page on Linear Recurrence Relations and Generating Functions

### 15 References

#### Key
- \(\text{book}\): a book
- \(\text{article}\): an article (paper) in an academic journal
- \(\text{link}\): a link to a web resource

The Fibonacci Quarterly journal: all papers older than 7 years are freely available online as PDFs; those published within the last 7 years are only available online to subscribers. Arranged in alphabetical order of author:


Art Benjamin and Jennifer Quinn have a wonderful knack of presenting proofs that involve counting arrangements of dominoes and tiling patterns in two ways that convince you that a formula really is true and not just "proved"! The identities proved mainly involve Fibonacci, Lucas and the General Fibonacci series with chapters on continued fractions, binomial identities, the Harmonic and Stirling numbers too. There is so much here to inspire students to find proofs for themselves and to show that proofs can be fun too!

Important notation difference: Benjamin and Quinn use \(f_n\) for the Fibonacci number \(F(n+1)\)

#### Bergum and Hoggatt (1975)

#### Benjamin, Carnes, Cloitre (2009)

#### Binet (1843)

#### Bro A Brousseau (1968)
A Sequence of Power Formulas The Fibonacci Quarterly vol 6 (1968) pages 81-83 as pdf

gives the recurrence relations of powers of Fibonacci's in terms of Fibonomials, as developed at the start of this page, but without explicitly stating the general formula and without recognizing the
Fibonomials.

Formulæ for F(n)/F(n-a), L(n)/F(n-a), L(n)/L(n-a) and G(a,b,n)/G(a,b,n-k) are developed into CFs

L E Dickson History of the Theory of Numbers: Vol 1 Divisibility and Primality
is a classic and monumental reference work on all aspects of Number Theory in 3 volumes (volume II is on Diophantine Analysis and volume III on Quadratic and Higher Forms). Although not up-to-date (the original edition was 1952) it is still a comprehensive summary of useful historical and early references on all aspects of Number Theory. The link is to a new cheap Dover paperback edition (2005) of Volume 1 which contains the most about Fibonacci Numbers, Lucas numbers and the golden section: see Chapter XVII on Recurring Series, Lucas' u_n, v_n, where he uses the series of Pisano for what we now call the Fibonacci numbers.

An introductory book strong on the geometry and natural aspects of the golden section but it does not include much on the mathematical detail. Beware - some of the formulæ in the Appendix are wrong! Dunlap has copied them from Vajda's book (see below) and he has faithfully preserved all of the original errors! The formulæ on this page (that you are now reading) are corrected versions and have been verified.

Fairgrieve and Gould (2005)


No - this is not a book about proportions of sand to cement 😊. The title is meant as an antidote to the "Abstract Mathematics" courses so often found in the curriculum of a university maths degree. As such, it is the book to dip into if you want to go really deeply into any part of the mathematics covered on this Fibonacci and Phi site. However, it quickly gets to an advanced mathematics undergraduate level after some nice introductions to every topic.
There are notes left in the margins which were inserted by students taking the original courses based on this book at Stanford university and they are interesting, often useful and sometimes quite funny.

Griffiths (2013)
Some errors and typos in the paper have been corrected on this webpage.

Hansen (1972)
"Generating Identities for Fibonacci and Lucas Triples" Rodney T Hansen, FQ (1972), pages 571-578 pdf

V E Hoggatt Jr "Fibonacci and Lucas Numbers" published by The Fibonacci Association, 1969 (Houghton Mifflin) (free online)
A very good introduction to the Fibonacci and Lucas Numbers written by a founder of the Fibonacci Quarterly.


Horadam (1967)
A F Horadam "Special Properties of the Sequence w_n(a,b;p,q)" FQ 5 (1967) pgs 424-434 pdf

Hudson and Winans (1981)
"A Complete Characterization of the Decimal Fractions That Can Be Represented as \( \sum 10^{k(a + 1)}F_{ai} \), where F_{ai} is the ai\textsuperscript{th} Fibonacci Number" R H Hudson, C F Winans The Fibonacci Quarterly 19, no. 5 (1981) pages 414-421. free pdf
See also


R Johnson (Durham university) has an excellent web page on the power of matrix methods to establish many Fibonacci formula with ease (but it does rely on at least undergraduate level matrix mathematics). See the Matrix methods for Fibonacci and Related Sequences link to a Postscript and PDF version on his Fibonacci Resources web page. The latest version (Nov 12, 2004) contains an appendix showing how formulae developed in Johnson's paper can prove almost all the identities here in my table above.


Koshy (2001) Fibonacci and Lucas Numbers with Applications, T Koshy, Wiley-Interscience, 2001, 648 pages. This book is packed full of an amazing number of Fibonacci and related equations, mostly culled from the pages of the Fibonacci Quarterly. Although initially impressive in its size and breadth, be aware that there are far too many typos, errors and missing or irrelevant conditions in many of its formulae as well as some glaring omissions and misattributions particularly with respect to the original references for a number of the formulae. Although Fibonacci representations of integers are included Zeckendorf himself is never even mentioned! There are lots of exercises with answers to the odd-numbered questions.


Lucas (1876) E Lucas, in Nuov. Corresp. Math. 2 (1876) , pages 74-75
See Dickson Vol 1 page 395


B Sharpe (1965) On Sums $F_x^2 \pm F_y^2$ Fib Quart (1965) 3.1 page 63 pdf

S Vajda, Fibonacci and Lucas numbers, and the Golden Section: Theory and Applications, Dover
This is a wonderful book, a classic, originally published in 1989 and now back in print in this Dover edition. This book is full of formulae on the Fibonacci numbers and Phi and the Lucas numbers. The whole book develops the formulae step by step, proving each from earlier ones or occasionally from scratch. It has a few errors in its formulae and all of them have been dutifully and exactly copied by authors such as Dunlap (see above) and others! Where I have identified errors, they have been corrected here on this page.

Vorob'ev (1951) Fibonacci Numbers N N Vorob'ev (2013 Dover paperback of the 1961 English version which itself was translated from the Russian 1951 edition)
An excellent and compact source book but only 66 pages long.